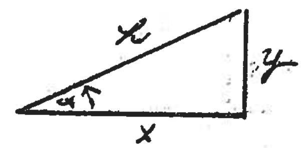
A SMATTERING OF CALCULUS

1. In a right triangle the tangent of an angle α is defined by

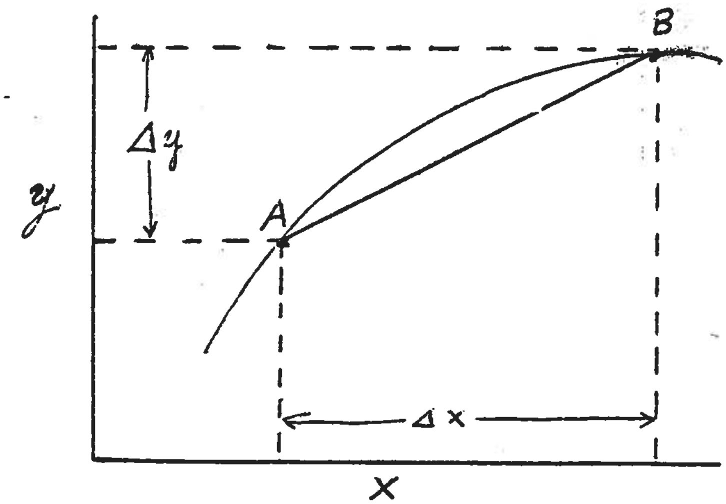
$$\tan \alpha \equiv \frac{y}{x}$$



This tangent is also called the slope of the line h. h is called the hypotenuse. (Note if α is between 90° and 180°, the tangent is a negative number).

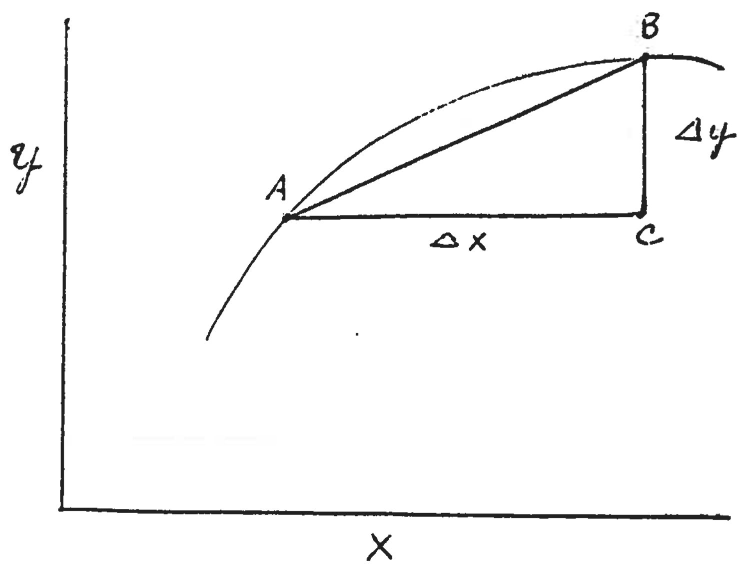
Definition of a derivative.

2. Consider the curve illustrated.



Draw a line connecting A and B. The projection ("shadow") of the line AB onto the x axis is called Δx. The projection of the line AB onto the y axis is called Δy.

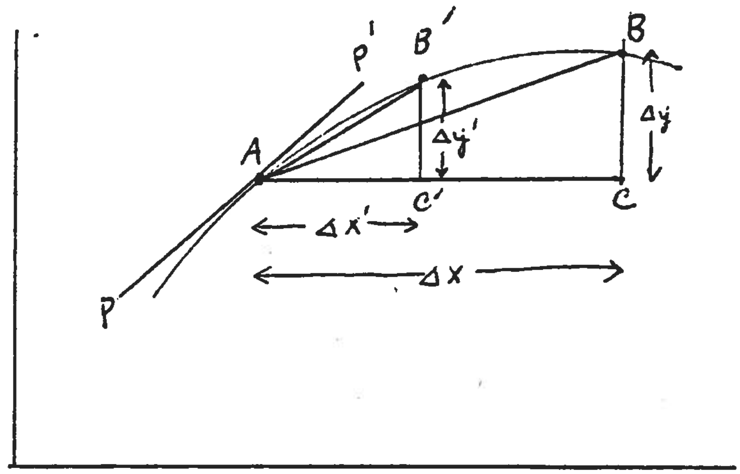
Construct the right triangle ABC with side AB (hypotenuse) and sides Δx and Δy as shown:



2. (continued)

   Then tan (∢ BAC) = $\frac{\Delta y}{\Delta x}$

   Now draw a line between the points AB' and construct a triangle AB'C' with sides Δx' and Δy'. Draw the line tangent to the curve at A indicating a segment of this line with the points P,P'.



For the line AB' the ratio $\frac{\Delta y'}{\Delta x'}$ is the tan ∢ B'AC'
As Δx' is made smaller the line AB' gets closer to being parallel to the segment PP' of the line tangent to curve at A. This statement is symbolically written as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \equiv \frac{dy}{dx}$$

where $\frac{dy}{dx}$ is the slope of the line tangent to the curve (In fact this line (and segment PP') is <u>defined</u> as having the same slope as the limiting case of AB'). $\frac{dy}{dx}$ is called a derivative.

Rule for finding derivatives

3. Consider the arbitrary curve of: y = f(x) where f denotes any function of x. The procedure for finding $\frac{dy}{dx}$ is as follows.

3. (continued)

Let x increase to $x + \Delta x$. Then y becomes $y + \Delta y$

$$y + \Delta y = f(x + \Delta x)$$

Subtract $y = f(x)$ from both sides

$$y + \Delta y - y = f(x + \Delta x) - f(x)$$

$$= \Delta y$$

Divide both sides by Δx

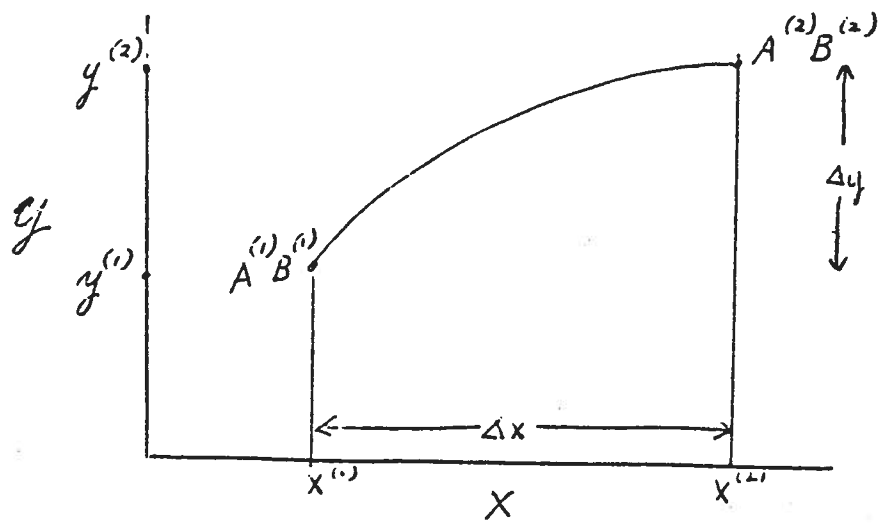
$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Find the limiting value of the ratio on the right hand side.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

<u>Subject: Rule for finding derivative of a product of two functions.</u>

4. Consider the product of two functions: $A(x)$ and $B(x)$



$$y(x) = A(x)B(x)$$

4. (continued)

Consider the change in y, A, B with x

$$y^{(1)} + \Delta y = y^{(2)} = A^{(2)}B^{(2)} = (A^{(1)} + \Delta A)(B^{(1)} + \Delta B)$$

$$= A^{(1)}B^{(1)} + (\Delta A)B^{(1)} + A^{(1)}(\Delta B) + \Delta A\Delta B$$

Subtract $\qquad y^{(1)} = A^{(1)}B^{(1)}$

$$\Delta y = (\Delta A)B^{(1)} + A^{(1)}(\Delta B) + \Delta A\Delta B$$

Divide by Δx

$$\frac{\Delta y}{\Delta x} = \frac{\Delta A}{\Delta x} B^{(1)} + A^{(1)}\frac{\Delta B}{\Delta x} + \frac{\Delta A\Delta B}{\Delta x}$$

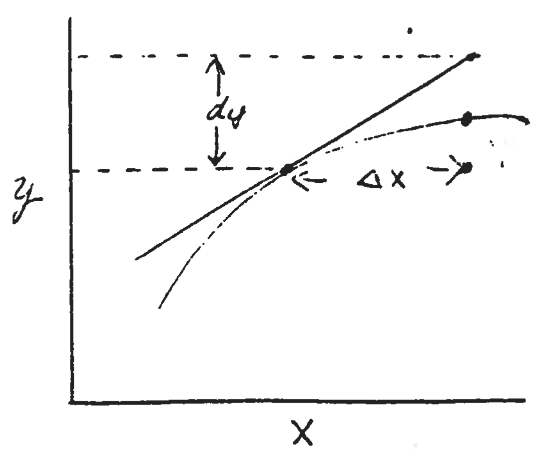
Note: as $\Delta x \to 0$, both ΔA and $\Delta B \to 0$. Their product approaches zero faster than either ΔA or ΔB alone. Another way of viewing this is to say that $\Delta A\Delta B$ is an "order of magnitude" smaller than either alone.
e.g. let ΔA or ΔB be $\sim 1/10$
Then $\Delta A\Delta B \sim 1/100$.
Thus $\Delta y \cong (\Delta A)B + A(\Delta B)$ and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{dA}{dx} B + \frac{AdB}{dx}$$

since $\Delta B \to 0$ and $\Delta A \to 0$.

Subject: differentials

5. For the line tangent to the curve we define the differential, dy.



Given $y = f(x)$

$$\frac{dy}{dx} = \frac{d(f(x))}{dx}$$

For an arbitrary Δx

$$dy \equiv \frac{dy}{dx}\Delta x$$

(definition)

5. (continued)

Since Δx is arbitrary, dy can have any value.

Let $f(x) = x$, $\frac{df(x)}{dx} = \frac{dx}{dx} = 1$

Then $dx \equiv 1 \cdot \Delta x$ by definition.

This special case assigns meaning to "dx". We can relate the differential of y to the differential of x of this special case.

$$dy = \frac{dy}{dx} dx$$

<u>Subject: natural logarithm</u>

6. Consider the special number e defined by the series

$$e^y \equiv 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots \frac{y^n}{n!} + \dots\dots$$

where $n!$ (n "factorial") is equal to $n(n-1)(n-2) \dots\dots \cdot 2 \cdot 1$

Suppose $v = e^y$ we say

$$\ln v = y$$

or $e^{\ln v} = v$ (natural logarithm) - of base e

i.e. any number v can be written as the number e with an exponent. That exponent is the "ln" of the original number v .

Compare: $100 = 10^2$

$\log_{10} 100 = 2$ $10^{\log_{10}(100)} = 100$

Note: $e^1 = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots \frac{1}{n!} + \dots$

7. Proof that $\frac{dv}{v} = d(\ln v)$

Apply the rule given in 3. to

$$y = \ln v$$

7. (continued)

$$y + \Delta y = \ln (v + \Delta v)$$

Subtract $\quad y = \ln v$

$$\Delta y = \ln (v + \Delta v) - \ln v$$

$$= \ln \left(\frac{v + \Delta v}{v}\right)$$

Aside: Note: Recall: $\frac{A^m}{A^n} = A^{m-n}$ Rule for exponents

and $\log_A A^m \equiv m$

$\log_A A^n \equiv n$

$\log_A A^{(m-n)} \equiv m-n$

Therefore $\log_A \frac{A^m}{A^n} = \log_A (A)^{m-n}$

$\equiv m-n$

$= \log_A A^m - \log_A A^n$

Recall: $\log A^m = m \log A$

$$\Delta y = \ln \left(1 + \frac{\Delta v}{v}\right)$$

Multiply the right hand side by $\frac{v}{v}$.

Divide both sides by Δv

$$\frac{\Delta y}{\Delta v} = \frac{v}{\Delta v} \cdot \frac{1}{v} \ln \left(1 + \frac{\Delta v}{v}\right)$$

$$= \frac{1}{v} \ln \left[\left(1 + \frac{\Delta v}{v}\right)^{\frac{v}{\Delta v}}\right]$$

Let $\frac{\Delta v}{v} = x$

$$\frac{\Delta y}{\Delta v} = \frac{1}{v} \ln \left[(1 + x)^{\frac{1}{x}}\right]$$

We shall now prove that

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

7. (continued)

Newton derived the formula for binomial expansion

$$(1 + X)^D = 1 + DX + \frac{D(D-1)X^2}{2!} + \frac{D(D-1)(D-2)X^3}{3!}$$

$$+ \ldots$$

This formula may be easily derived for positive integer values of D using mathematical induction. If D is negative or a fraction the expression becomes an infinite series. This infinite series is only meaningful if it <u>converges</u> to a number.

Consider $(1 + x)^{\frac{y}{x}}$. Use Newton's Formula.

$$(1 + x)^{\frac{y}{x}} = 1 + \frac{yx}{x} + \left(\frac{y}{x}\right)\left(\frac{y}{x} - 1\right)\frac{x^2}{2!} + \left(\frac{y}{x}\right)\left(\frac{y}{x} - 1\right)\left(\frac{y}{x} - 2\right)\frac{x^3}{3!} + \ldots$$

$$= 1 + y + \frac{y^2}{2!} - \frac{1}{2!}\,yx + \frac{y^3}{3!} - \frac{2}{3!}\,y^2x - \frac{3y^2x}{3!} + \frac{6x^2y}{3!} + \ldots .$$

$$\lim_{x \to 0} (1 + x)^{\frac{y}{x}} = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$$

$$= e^y$$

If, in particular $y = 1$

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e^1 = e$$

Thus $\lim\limits_{\Delta v \to 0} \frac{\Delta v}{v} = \lim\limits_{x \to 0} x = 0$ and

$$\lim_{\Delta v \to 0} \frac{\Delta y}{\Delta v} \equiv \frac{dy}{dv} = \frac{1}{v} \ln e = \frac{1}{v}$$

Return to the definition of a differential (4).

7. (continued)

$$dy = \left(\frac{dy}{dx}\right)dx \qquad \text{or}$$

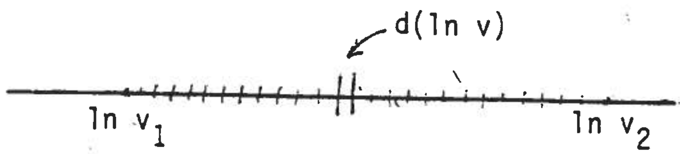
$$dy = \left(\frac{dy}{dv}\right)dv$$

$$e^{\ln v} = v$$

$$d(\ln v) = \left(\frac{dy}{dv}\right)dv = \frac{1}{v}\,dv$$

This formula will be used in the study of thermodynamics.

8. If the differential represents a tiny segment on a line, then adding all of the tiny segments gives a larger segment between 2 points



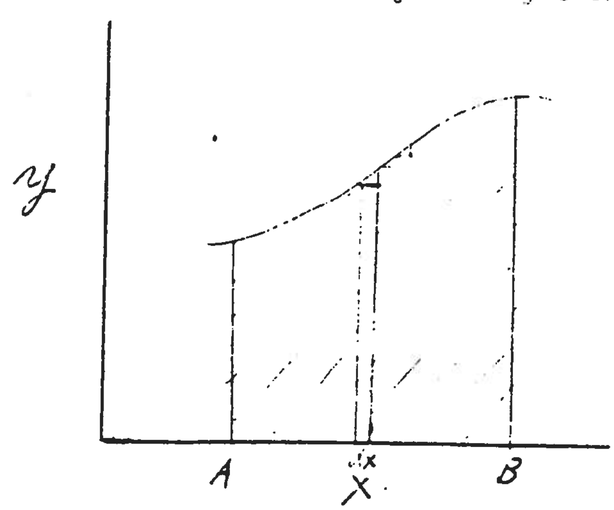
⟵ length = $\ln v_2 - \ln v_1$ ⟶

This addition is symbolized by $\int_{\ln v_1}^{\ln v_2} d\,(\ln v) = \ln v_2 - \ln v_1$

9. The symbol, invented by Leibnitz,

$$\int_{X=A}^{X=B} y\,dx$$

represents the area under the arbitrary curve y shown below:



9. (continued)

Each little segment ydx is the area of a rectangle of height y and width dx. The sum of all of the little rectangles is the area under the curve. ($\int$ is the German "s")
Leibnitz also used the notation:

$$\text{Area} = \lim_{\substack{\Delta x_i \to 0 \\ n\to\infty}} \sum_{i=1}^{n} y_i \Delta x_i \quad \text{where} \quad \sum_{i=1}^{n} \Delta x_i = B-A$$

where $\sum_{i=1}^{n}$ means "the sum over all segments labelled with the index i, where i takes on integer values from 1 to n."

Obviously as the segments Δx_i become thinner more of them can fit between A and B so that as $\Delta x_i \to 0$, $n \to \infty$.

10. The differential of a product

$y(x) = A(x)B(x)$

Use the rule given in 4.

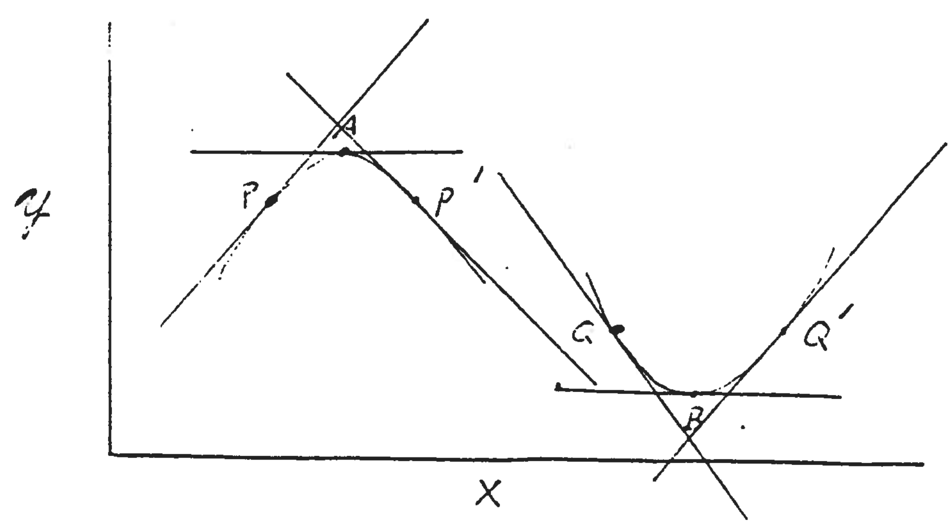
$$dy = \frac{d(AB)dx}{dx}$$

$$= \underbrace{\frac{AdBdx}{dx}} + \underbrace{\frac{BdAdx}{dx}}$$

$$d(AB) = AdB + BdA$$

This formula is used in thermodynamics.

11.



11. (continued)

Point A is called a <u>maximum</u> point on the curve. Point B is called a <u>minimum</u>. At both points the differential dy=0 and the slope of the tangent $= \frac{dy}{dx} = 0$.

Note that around a maximum the slope is positive at P, zero at A, and negative at P' (i.e. continuously decreasing)
Around a minimum the slope is negative at Q, zero at B, positive at Q' (i.e. continuously increasing).

12. <u>Differential of a function of several variables:</u>
Consider a function U which is dependent on three variables x, y, z.

$$\Delta U = U_{final} - U_{initial} = U(x+\Delta x,\ y+\Delta y,\ z+\Delta z) - U(x,\ y,\ z)$$

This can be rewritten as

$$\Delta U = U(x+\Delta x,\ y+\Delta y,\ z+\Delta z) - U(x,\ y+\Delta y,\ z+\Delta z)$$
$$+\ U(x,\ y+\Delta y,\ z+\Delta z) - U(x,\ y,\ z+\Delta z)$$
$$+\ U(x,\ y,\ z+\Delta z) - U(x,\ y,\ z)$$
$$= \frac{(U(x+\Delta x),\ y+\Delta y,\ z+\Delta z) - U(x,\ y+\Delta y,\ z+\Delta z)}{\Delta x}\Delta x$$
$$+ \frac{(U(x,\ y+\Delta y,\ z+\Delta z) - U(x,\ y,\ z+\Delta z))}{\Delta y}\Delta y$$
$$+ \frac{(U(x,\ y,\ z+\Delta z) - U(x,\ y,\ z))}{\Delta z}\Delta z$$

Definitions:

$$\lim_{\Delta x \to 0} \frac{(U(x+\Delta x,\ y+\Delta y,\ z+\Delta z) - U(x,\ y+\Delta y,\ z+\Delta z))}{\Delta x} \equiv \left(\frac{\partial U}{\partial x}\right)_{y,z}$$

$$\lim_{\Delta y \to 0} \frac{(U(x,\ y+\Delta y,\ z+\Delta z) - U(x,\ y,\ z+\Delta z))}{\Delta y} \equiv \left(\frac{\partial U}{\partial y}\right)_{x,z}$$

$$\lim_{\Delta z \to 0} \frac{(U(x,\ y,\ z+\Delta z) - U(x,\ y,\ z))}{\Delta z} \equiv \left(\frac{\partial U}{\partial z}\right)_{x,y}$$

12. (continued)

Recall definition of differential:

$$dy \equiv \left(\frac{dy}{dx}\right)\Delta x \qquad \text{and } dx \equiv \Delta x$$

so that $dy = \left(\frac{dy}{dx}\right)dx$

Similarly

$$du \equiv \left(\frac{\partial U}{\partial x}\right)_{y,z}dx + \left(\frac{\partial U}{\partial y}\right)_{x,z}dy + \left(\frac{\partial U}{\partial z}\right)_{x,y}dz$$

$\left(\frac{\partial U}{\partial x}\right)_{y,z}$ is known as "partial derivative of U with respect to x"