Tentative syllabus for 110.441, CALCULUS ON MANIFOLDS

A **manifold** is a space which looks locally, but not necessarily globally, like Euclidean space: the

`https : //en.wikipedia.org/wiki/Exhaust_manifold`

under the hood of a car is a good example, as is the neighborhood of a neutron star or black hole. This course is a kind of continuation of the Calc I - III sequence, extending the methods introduced there to higher dimensions and curved spaces.

Calc III and a good knowledge of linear algebra are prerequisites. Some background in analysis, topology, or differential geometry is useful but not essential. In the latter part of the course students will be asked to present standard material from Spivak's text in class, but the first part of the course will be devoted to more geometric material. Here is a tentative summary of the topics to be covered:

§I Vector calculus revisited

1.0 Open subspaces $\mathcal{O} \subset \mathbb{R}^n$ of Euclidean space, and smooth maps

$$\mathbb{R}^m \supset \mathcal{O}' \xrightarrow{F} \mathcal{O} \subset \mathbb{R}^n$$

between them. Tangent ($T(\mathcal{O}) = \mathcal{O} \times \mathbb{R}^n$) and cotangent ($T^*(\mathcal{O}) = \mathcal{O} \times \mathbb{R}^{n*}$ (alternately: phase space)) bundles as parametrized families of spaces of first-order linear differential operators, and the derivative $T(F) : T(\mathcal{O}') \to T(\mathcal{O})$ as a map of bundles of vector spaces. The chain rule

$$T(G \circ F) = T(G) \circ T(F) : T(\mathcal{O}') \to T(\mathcal{O}) \to T(\mathcal{O}'') \ .$$

1.1 The inverse and implicit function theorems (eg F (smooth) has an inverse near $x \in \mathcal{O}'$ if $T_x(F) : T_x(\mathcal{O}') \to T_{F(x)}(\mathcal{O})$ is invertible (ex $\log : \mathbb{R} \to \mathbb{R}$ near $x = 1$). **Generic** properties of smooth maps; statement (but not proof) of Sard's theorem.

§II Coordinate charts; manifolds as locally Euclidean spaces

Definition & Examples (of manifolds and their tangent (etc) bundles). The Riemann sphere and its generalizations. Real and complex projective spaces ($P_n(\mathbb{R}) \subset P_n(\mathbb{C})$); Riemann surfaces in $P_2(\mathbb{C})$. Possible extra topics: groups and group actions; Morse's lemma; the Schwarzschild solution.

§III Transversality

When is the intersection of two (smooth) submanifolds again a (smooth) submanifold (ex $\mathbb{R}^j, \mathbb{R}^k \subset \mathbb{R}^n$)? Transversality of intersections, and of maps. [This is all just a reformulation of the implicit function theorem in the language of tangent bundles.]

§IV Differential forms and vector bundles, (following Spivak)

Tensor, symmetric, and exterior algebras. Partitions of unity. Vector bundles in general, and their spaces of sections. Differential forms and covariant derivatives. Orientations, volumes, and integration on manifolds. Poincare's lemma, de Rham cohomology, and Maxwell's equations.

Some References:

1) Any good vector calculus text, eg Marsden & Tromba

2) M Spivak, **Calculus on manifolds** [I plan to have a scan available]

3) J Milnor, **Topology from a differentiable viewpoint**, available at

`http : //www.maths.ed.ac.uk/ ~ aar/papers/milnortop.pdf`