Tentative syllabus for 110.441 (Fall 017), CALCULUS ON MANIFOLDS

A **manifold** is a space which looks locally, but not necessarily globally, like Euclidean space: the

`https://en.wikipedia.org/wiki/Exhaust_manifold`

under the hood of a car is a good example, as is the neighborhood of a neutron star or black hole. This course is a kind of continuation of the Calc I - III sequence, extending the methods introduced there to higher dimensions and curved spaces.

Calc III and a good knowledge of linear algebra are the main prerequisitesfor this course; some knowledge of topology and measure theory would be helpful. Spivak's *Calculus on Manifolds* will be the background text for the first half; I'll distribute a scanned copy FYI. With these tools we can analyze Maxwell's equations in modern terms, as a gauge theory with the circle as its gauge group.

In previous years there has been enough time later in the course to extend these ideas to (classical, not quantum!) **non-abelian** gauge theories

`https://en.wikipedia.org/wiki/Gauge_theory`

`https://en.wikipedia.org/wiki/Yang_Mills_theory`

This involves generalizing ideaa from the first part of the course to bundles more general than the tangent bundle; this involves some examples from the general theory of Lie groups. The payoff is a solution to the problem of 'action at a distance', which goes back to Newton and before.

§I Vector calculus and differential forms

1.0 Open subspaces $\mathcal{O} \subset \mathbb{R}^n$ of Euclidean space, and smooth maps

$$\mathbb{R}^m \supset \mathcal{O}' \xrightarrow{F} \mathcal{O} \subset \mathbb{R}^n$$

between them. Tangent ($T(\mathcal{O}) = \mathcal{O} \times \mathbb{R}^n$) and cotangent ($T^*(\mathcal{O}) = \mathcal{O} \times \mathbb{R}^{n*}$ (alternately: phase space)) bundles as parametrized families of spaces of first-order linear differential operators, and the derivative $T(F) : T(\mathcal{O}') \to T(\mathcal{O})$ as a map of bundles of vector spaces. The chain rule

$$T(G \circ F) = T(G) \circ T(F) : T(\mathcal{O}') \to T(\mathcal{O}) \to T(\mathcal{O}'') \ .$$

1.1 The inverse and implicit function theorems (eg F (smooth) has an inverse near $x \in \mathcal{O}'$ if $T_x(F) : T_x(\mathcal{O}') \to T_{F(x)}(\mathcal{O})$ is invertible (ex $\log : \mathbb{R} \to \mathbb{R}$ near $x = 1$). **Generic** properties of smooth maps; statement (but not proof) of Sard's theorem.

1.2 Coordinate charts; manifolds as locally Euclidean spaces

Definitions & Examples (of manifolds and their tangent (etc) bundles). The Riemann sphere and its generalizations. Real and complex projective spaces ($P_n(\mathbb{R}) \subset P_n(\mathbb{C})$); Riemann surfaces in $P_2(\mathbb{C})$. Possible extra topics: groups and group actions; Morse's lemma; the Schwarzschild solution.

1.3 Transversality

When is the intersection of two (smooth) submanifolds again a (smooth) submanifold (ex $\mathbb{R}^j, \mathbb{R}^k \subset \mathbb{R}^n$)? Transversality of intersections, and of maps. [This is all just a reformulation of the implicit function theorem in the language of tangent bundles.]

1.4 Differential forms and vector bundles (following Spivak)

Tensor, symmetric, and exterior algebras. Partitions of unity. Vector bundles in general, and their spaces of sections. Differential forms and covariant derivatives. Orientations, volumes, and integration on manifolds. Poincare's lemma, de Rham cohomology, and Maxwell's equations.

§II Applictions to modern physics

Maxwell's equations, in terms of differential forms; their solutions in terms of cohomology classes. Gauge equivalence.

Non-abelian generalizations: principal bundles, connections (ie covariant differentiation) on them; gauge equivalence and spaces of connections. Curvature forms and the Yang-Mills action functional.

Some References:

1) Any good vector calculus text, eg Marsden & Tromba's **Vector Calculus**

2) M Spivak, **Calculus on manifolds** [I plan to have a scan available]

3) J Milnor, **Topology from a differentiable viewpoint**, available at

`http : //www.maths.ed.ac.uk/` $\sim$ `aar/papers/milnortop.pdf`

4) Some overly detailed notes

www.math.mcgill.ca/gantumur/math581w12/downloads/picard$_{\mathtt{YM}}$.pdf

on Yang-Mills theory, which has some accessible parts.