Tentative syllabus for 110.441 (Fall 017), Calculus on Manifolds

A manifold is a space which looks locally, but not necessarily globally, like Euclidean space: the

https://en.wikipedia.org/wiki/Exhaust_manifold

under the hood of a car is a good example, as is the neighborhood of a neutron star or black hole. This course is a kind of continuation of the Calc I - III sequence, extending the methods introduced there to higher dimensions and curved spaces.

Calc III and a good knowledge of linear algebra are the main prerequisites for this course; some knowledge of topology and measure theory would be helpful. Spivak’s Calculus on Manifolds will be the background text for the first half; I’ll distribute a scanned copy FYI. With these tools we can analyze Maxwell’s equations in modern terms, as a gauge theory with the circle as its gauge group.

In previous years there has been enough time later in the course to extend these ideas to (classical, not quantum!) non-abelian gauge theories

https://en.wikipedia.org/wiki/Gauge_theory


This involves generalizing ideas from the first part of the course to bundles more general than the tangent bundle; this involves some examples from the general theory of Lie groups. The payoff is a solution to the problem of ‘action at a distance’, which goes back to Newton and before.

§I Vector calculus and differential forms

1.0 Open subspaces \( O \subset \mathbb{R}^n \) of Euclidean space, and smooth maps

\[
\mathbb{R}^m \supset O' \xrightarrow{F} O \subset \mathbb{R}^n
\]

between them. Tangent \((T(O) = O \times \mathbb{R}^n)\) and cotangent \((T^*(O) = O \times \mathbb{R}^n^*)\) bundles as parametrized families of spaces of first-order linear differential operators, and the derivative \(T(F) : T(O') \rightarrow T(O)\) as a map of bundles of vector spaces. The chain rule

\[
T(G \circ F) = T(G) \circ T(F) : T(O') \rightarrow T(O) \rightarrow T(O'').
\]

1.1 The inverse and implicit function theorems (eg \( F \) (smooth) has an inverse near \( x \in O' \) if \( T_x(F) : T_x(O') \rightarrow T_{F(x)}(O) \) is invertible (ex \( \log : \mathbb{R} \rightarrow \mathbb{R} \) near \( x = 1 \)). Generic properties of smooth maps; statement (but not proof) of Sard’s theorem.
1.2 Coordinate charts; manifolds as locally Euclidean spaces

Definitions & Examples (of manifolds and their tangent (etc) bundles). The Riemann sphere and its generalizations. Real and complex projective spaces \( P_n(\mathbb{R}) \subset P_n(\mathbb{C}) \); Riemann surfaces in \( P_2(\mathbb{C}) \). Possible extra topics: groups and group actions; Morse’s lemma; the Schwarzschild solution.

1.3 Transversality

When is the intersection of two (smooth) submanifolds again a (smooth) submanifold (ex \( \mathbb{R}^j, \mathbb{R}^k \subset \mathbb{R}^n \))? Transversality of intersections, and of maps. [This is all just a reformulation of the implicit function theorem in the language of tangent bundles.]

1.4 Differential forms and vector bundles (following Spivak)


§II Applications to modern physics

Maxwell’s equations, in terms of differential forms; their solutions in terms of cohomology classes. Gauge equivalence.

Non-abelian generalizations: principal bundles, connections (ie covariant differentiation) on them; gauge equivalence and spaces of connections. Curvature forms and the Yang-Mills action functional.

Some References:

1) Any good vector calculus text, eg Marsden & Tromba’s Vector Calculus
2) M Spivak, Calculus on manifolds [I plan to have a scan available]
3) J Milnor, Topology from a differentiable viewpoint, available at

   \[ \text{http://www.maths.ed.ac.uk/~aar/papers/milnortop.pdf} \]

4) Some overly detailed notes

   \[ \text{www.math.mcgill.ca/gantumur/math581w12/downloads/picardYM.pdf} \]

on Yang-Mills theory, which has some accessible parts.