WORKSHOP IN GEOMETRIC MEASURE THEORY
JOHNS HOPKINS UNIVERSITY
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Robert Hardt, *Rice University.*

One Dimensional Rectifiable Varifolds and Some Applications

Varifolds were originally introduced to describe various 2 dimensional minimal surfaces and soap film models. A stationary one-dimensional varifold roughly models a spider-web (possibly of variable thickness). With a suitable lower density bound, a stationary one dimensional rectifiable varifold enjoys a regularity property due to F.Almgren and W.Allard (1976). Such 1d varifolds are useful in describing various ramified optimal transport problems. Another application uses pairs of 1d varifolds to model Michel trusses. These are cost optimal 1d balanced structures consisting of bars and cables. Introduced by Michel in 1904, they have been treated in the Mechanical Engineering literature and in interesting papers by R.Kohn and G. Strang (1983) and by G.Bouchitte, W.Gangbo, and P.Seppecher (2008). There are many basic open questions about the location and structure of Michel trusses. The varifold model allows one to consider associated evolution and higher dimensional problems.

Andrew Lorent, *University of Cincinnati.*

Null Lagrangian measures

Compensated compactness is an important method used to solve nonlinear PDEs, in particular in the study of hyperbolic conservation laws. One of the simplest formulations of a compensated compactness problem is to ask for conditions on a compact set $\mathcal{K} \subseteq M^{m\times n}$ such that

$$\lim_{j\to\infty} \|\mathrm{dist}(Du_j,\mathcal{K})\|_{L^p} = 0 \text{ and } \sup_j \|u_j\|_{W^{1,p}} < \infty \ \Rightarrow \{Du_j\}_j \text{ is precompact in } L^p. \tag{1}$$

Let $M_1, M_2, \ldots, M_q$ denote the set of all minors of $M^{m\times n}$. A sufficient condition for (1) is that any probability measure μ supported on $\mathcal{K}$ satisfying

$$\int M_k(X)d\mu(X) = M_k\left(\int X d\mu(X)\right) \text{ for all } k \tag{2}$$

is a Dirac measure. We call measures that satisfy (2) **Null Lagrangian Measures** and in the following we denote the set of Null Lagrangian Measures supported on $\mathcal{K}$ by $\mathcal{M}^{pc}(\mathcal{K})$. For general m, n, a necessary and sufficient condition for triviality of $\mathcal{M}^{pc}(\mathcal{K})$ was an open question even in the case where $\mathcal{K}$ is a linear subspace of $M^{m\times n}$. We answer this question and provide a necessary and sufficient condition for any linear subspace $\mathcal{K} \subseteq M^{m\times n}$. The ideas also allow us to show that for any $d \in \{1,2,3\}$, d-dimensional subspaces $\mathcal{K} \subseteq M^{m\times n}$ support non-trivial Null Lagrangian Measures if and only if $\mathcal{K}$ has Rank-1 connections. This is known to be false for $d \geq 4$.

Null Lagrangian measures also arise in the study of certain nonlinear PDEs that are naturally associated to a set K in the space of matrices, and triviality of $M^{pc}(K)$, often leads to nice properties of the PDEs. A particular example comes from the method of compensated compactness, in which the Young measures can be viewed as Null Lagrangian measures. The ideas developed allow us to answer a question raised by Kirchheim, Müller and Sverák on the structure of $M^{pc}(K)$ for some nonlinear submanifold $K \subseteq \mathbb{R}^{3\times 2}$ that is associated to a well known 2×2 system of conservation laws with one entropy/entropy flux pair. This is joint work with Guanying Peng.

Chris Miller, *Ohio State University.*

Volumetric tameness in structures on the real fields

The notion of "structure (on the real field)" arises from model theory, a branch of mathematical logic, but it is easy to communicate a precise definition to a general mathematical audience. Arguably, the best-behaved structures are the o-minimal ones, that is, those for which every set belonging to the stucture has only finitely many connected components. Over the last two and a half decades, o-minimality (the study of o-minimal structures) has proven to be of considerable interest to analytic geometers. I am currently studying structures having the property that the locally C^0behavior of sets in the structure is no worse than the locally Lipschitz behavior. I will define precisely what I mean by this, which leads to notions of "volumetric tameness" of structures. I will illustrate that : (a) there exist interesting structures that are volumetrically tame but not o-minimal ; and (b) there exist structures that are not volumetrically tame, yet are nevertheless regarded as geometrically tame in certain precise senses.

Gareth Speight, *University of Cincinnati.*

Geometry and Differentiability in Carnot Groups

Carnot groups are non-Euclidean spaces which nevertheless admit translations, dilations and a distance measured using a rich family of curves. We discuss how the geometry of such spaces affects differentiability through two problems. The first is the problem of generalizing and refining Rademacher's theorem on almost everywhere differentiability of Lipschitz functions. The second is the problem of Whitney extension, which extends a map defined on a subset to a map defined on the whole space with desired differentiability properties.

Monica Torres, *Purdue University.*

Divergence-measure fields : Gauss-Green formulas and normal traces

The Gauss-Green formula is a fundamental tool in analysis. In this talk we present new Gauss-Green formulas for divergence-measure fields (i.e. ; vector fields in L^p, $1 \leqslant p \leqslant \infty$ whose divergence is a Radon measure) which hold on rough open sets of finite perimeter or general open sets. This is a joint work with Gui-Qiang Chen (University of Oxford), Giovanni Comi (Scuola Normale Superiore, Pisa) and Qinfeng Li (University of Texas, San Antonio).

Changyou Wang, *Purdue University.*

Regularity of absolute minimizers for continuous convex Hamiltonians

For any $n \geqslant 2$, $\Omega \subseteq \mathbb{R}^n$, and any given convex and coercive Hamiltonian function $H \in C^0(\mathbb{R}^n)$, we find an optimal sufficient condition on H, that is, for any $c \in \mathbb{R}$, the level set $H^{-1}(c)$ does not contains any line segment, such then any absolute minimizer $u \in AM_H(\Omega)$ enjoys the linear approximation property. As consequences, we show that when $n = 2$, if $u \in AM_H(\Omega)$ then $u \in C^1$; and if $u \in AM_H(\mathbb{R}^2)$ satisfies a linear growth at the infinity, then u is a linear function on $\mathbb{R}^2$. In particular, if H is a strictly convex Banach norm $\|\cdot\|$ on $\mathbb{R}^2$, e.g. the l_α-norm for $1 < \alpha < 1$, then any $u \in AM_H(\Omega)$ is C^1. The ideas of proof are, instead of PDE approaches, purely variational and geometric. This is a joint work with Feng Ya, and Yuan Zhou.

Neshan Wickramasekera, *University of Cambridge.*

Uniqueness of blow-ups and asymptotic decay for Dirichlet energy minimizing multi-valued functions

In the early 1980's Almgren developed a theory of Dirichlet energy minimizing multi-valued functions, proving that the Hausdorff dimension of the singular set (including branch points) of such a function is at most $n-2$ where n is the dimension of its domain. Almgren used this result in an essential way to show that the same upper bound holds for the dimension of the singular set of an area minimizing n dimensional rectifiable current of arbitrary codimension. In either case, the dimension bound is sharp. I will describe recent work (joint with Brian Krummel) which develops estimates to study the asymptotic behaviour of a Dirichlet energy minimizing q valued function on approach to its branch set. Our estimates imply that a Dirichlet energy minimizer at a.e. point along its branch set (with respect to the $n-2$ dimensional Hausdorff measure) has a unique set of homogeneous multi-valued cylindrical tangent functions (blow-ups) to which the minimizer, modulo a set of single-valued harmonic functions, decays exponentially fast upon rescaling. A corollary is that the branch set is countably $n-2$ rectifiable. (The corollary, without the uniqueness result, was also established recently by De Lellis–Marchese–Spadaro–Valtorta using a different method due to Naber–Valtorta). This work generalizes our earlier work in which we treated the special case $q = 2$; the general case requires some different and new ideas which I shall outline.