Graduate diagnostic test section I
Quantum Mechanics & Electromagnetism

August 24, 2012, 9 am-noon.

This a three-hour closed book diagnostic test. Please show us what you can do with the assistance only of the attached formula sheet. Ask your proctor for clarification if the text is unclear. Provide detailed reasoning using text, sketches and equations and express your answers using variables defined in the problem. Describe your ideas for partial credit even if you cannot solve the problem.

Problem 1

A particle of mass m is confined to a two-dimensional square box with sides of length a: $0 \leq x \leq a, 0 \leq y \leq a$. Thus the potential energy is

$$V_{\text{square box}}(x, y) = \begin{cases} 0, & 0 \leq x \leq a, \quad 0 \leq y \leq a \\ \infty, & \text{otherwise.} \end{cases}$$

(a) Determine the eigenfunctions and corresponding energy levels of the particle. What is the degeneracy of each level?

(b) A small perturbing potential $V_\lambda = \lambda\delta(x - y)$, which is non-zero just along the diagonal of the box is introduced. Working in lowest order in perturbation theory determine the energy eigenvalues and the corresponding zeroth order eigenstates.

(The following integral can be helpful: $\int_0^\pi \sin^2 nz \sin^2 mz dz = \frac{\pi}{4}(1 + \frac{1}{2}\delta_{n,m})$, where n and m are positive integers.)

Problem 2

The Hamiltonian of a rigid rotor in a homogeneous magnetic field is given by

$$\hat{H} = \frac{1}{2I}\hat{\mathbf{L}}^2 + \omega\hat{L}_z.$$

(a) What are the energies and corresponding wave functions of the stationary states of the rotor?

(b) At $t = 0$ the rotor is in the following state written in position space representation:

$$< \theta\phi|\psi(t=0) >= N \sin\theta \sin\phi.$$

Determine the normalization (N). Compute the expectation value of the energy and of $\hat{L}_x$.

(c) Give the state at $t > 0$.

Problem 3

Two particles each of mass m are bound in a one dimensional harmonic oscillator potential and interact with each other through an attractive force. Hence the Hamiltonian is

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + \frac{1}{2}k(\hat{x}_1^2 + \hat{x}_2^2) + \frac{1}{2}K(\hat{x}_1 - \hat{x}_2)^2.$$

(a) What are the energies and eigenkets of this system? Identify the four lowest energy states if $K = \frac{k}{16}$.
(*Hint:* Consider rewriting the Hamiltonian using the following linear combinations of x_1 and x_2:

$$\zeta = \frac{1}{\sqrt{2}}(x_1 + x_2), \quad \eta = \frac{1}{\sqrt{2}}(x_1 - x_2).)$$

(b) Note that the one dimensional harmonic oscillator stationary states have the following property under parity:

$$\phi_n(x) =< x|n >= (-1)^n < -x|n >= (-1)^n \phi_n(-x).$$

i. Assume that the two particles are identical zero-spin bosons. Write the total wave function for the three lowest energy states.

ii. Assume that the two particles are spin $\frac{1}{2}$ fermions. Which are the three lowest energy states?

Problem 4

Consider an isolated magnetized sphere of radius R with a uniform magnetization M_o. Find the **B** field and **H** field everywhere.

Problem 5

Electric potential values on two concentric spherical surfaces of radii $r = a$ and $r = b$ ($b > a$) are $V_0 \cos^2\theta$ and 0 respectively, where V_o is a constant and θ is the angle measured from an axis through the center of the spheres. Find the potential everywhere in the empty space between the two spheres.

Problem 6

A plane $z = 0$ separates vacuum ($z < 0$) and a dielectric ($z > 0$). The electric field of a plane electromagnetic (EM) wave propagating in vacuum with electric permittivity ε_o and magnetic permeability μ_o has the form

$$\boldsymbol{E} = A\cos(\omega t - 4bx - 3bz)\boldsymbol{j},$$

where A and b are constants in units of V/m and m^{-1}, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are unit vectors along the Cartesian coordinates x, y, z and t is time. This EM wave is incident on the dielectric with permittivity $\varepsilon = 2.25\varepsilon_o$ and permeability $\mu = \mu_o$.

(a) Describe the polarization of the EM wave on the vacuum side.
(b) What may be the values of the angular frequency ω and wavelength λ of the EM wave on the vacuum side?
(c) Find the direction and polarization of the transmitted EM wave within the dielectric.

Formulae

One-dimensional harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2;$$

$$[\hat{x}, \hat{p}] = i\hbar, \quad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p}), \quad [\hat{a}, \hat{a}^\dagger] = 1$$

$$\hat{H}|n >= \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})|n >= \hbar\omega(n + \frac{1}{2})|n >, \quad n = 0, 1, 2, ...$$

$$|n >= \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0 >$$

Angular momentum and spherical harmonics

(i, j, k are the Cartesian components of the angular momentum, also denoted as x, y, z.)

$$[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k, \quad i, j, k = 1, 2, 3. \quad \hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2$$

$$\hat{J}_\pm|j, m >= \hbar\sqrt{j(j+1) - m(m \pm 1)}|j, m \pm 1 >$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta Y^*_{l,m}(\theta, \phi)Y_{l',m'}(\theta\phi) = \delta_{l,l'}\delta_{m,m'}$$

$$Y_{0,0} = \sqrt{\frac{1}{4\pi}}, \qquad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}, \qquad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

Useful Equations

- Maxwell's Equations:

$$\begin{array}{ll} \nabla \cdot \vec{D} = \rho(\vec{x}) & \oint \vec{D} \cdot d\vec{a} = Q_{enclosed} \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} & \oint \vec{E} \cdot d\vec{\ell} = -\frac{\partial \Phi_B}{\partial t} \\ \nabla \cdot \vec{B} = 0 & \oint \vec{B} \cdot d\vec{a} = 0 \\ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} & \oint \vec{H} \cdot d\vec{\ell} = I_{enclosed} + \frac{\partial \Phi_D}{\partial t} \end{array}$$

- Boundary Conditions: $\hat{n}$ is normal to the plane boundary, σ and $\vec{K}$ are surface charge and current densities.

$$\begin{array}{ll} \hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \sigma & \hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0 \\ \hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0 & \hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K} \end{array}$$

- Static Potentials: φ and $\vec{A}$ are the electric scalar and magnetic vector potentials,

$$\vec{E} = -\nabla\varphi \quad \vec{H} = \nabla \times \vec{A}$$

- Potential of a dipole:

$$\varphi(\vec{r}) = \frac{\vec{P} \cdot \vec{r}}{4\pi\epsilon_0 r^3}$$

- Dipole Moment:

$$\vec{P} = \int \vec{x}\rho(\vec{x}) d^3x$$

- Lorentz Force:

$$\frac{d\vec{p}}{dt} = q\vec{E} + q\vec{v} \times \vec{B}$$

- Homogeneous Media:

$$\begin{array}{l} \vec{D} = \epsilon\vec{E} = \epsilon_0\vec{E} + \vec{P} \\ \vec{B} = \mu\vec{H} = \mu_0\left(\vec{H} + \vec{M}\right) \end{array}$$

- Definitions:
  - Capacitance - $C = Q/V$
  - Resistance - $V = IR$
  - Inductance - $V = -LdI/dt$

Electric dipole

$$V_{dipole} = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}$$

$$\mathbb{E}_{dipole} = \frac{1}{4\pi\varepsilon_0} \frac{p}{r^3} [2\cos\theta\, \hat{\mathbf{r}}_1 + \sin\theta\, \hat{\theta}_1]$$

For a dielectric with polarization $\mathbf{P}$, the potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0} \int \frac{-\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' + \frac{1}{4\pi\varepsilon_0} \int \frac{\mathbf{P}(\mathbf{r}') \cdot \mathbf{n}_1}{|\mathbf{r} - \mathbf{r}'|} da'$$

$$\mathbf{D} = \varepsilon_0 \mathbb{E} + \mathbf{P}$$

Quadrupole moment tensor : $Q_{ij} = \int (3\, x_i x_j - r^2 \delta_{ij})\, \rho(r)\, d\tau$

Current density : $\mathbf{J} = n\, q\, v = \rho\, v$

Ohm's law : $\mathbf{J} = \sigma\, \mathbb{E}$

Capacitor : $Q = CV$

Continuity Equation : $\nabla \cdot \mathbf{J} + \frac{\partial \rho_f}{\partial t} = 0,$

Biot-Savart Law

$$B(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{I\, d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_f(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau'$$

Vector potential

$$A(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_f(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau'$$

$$d\vec{A}(r) = \frac{\mu_0}{4\pi} \frac{d\vec{I}}{r}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla V$$

Some useful mathematical information:

1. General solution of $\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial V}{\partial\theta} = 0$ in spherical coordinates

$$V(r,\theta) = \sum_{n=0}^{\infty}(A_n r^n + \frac{B_n}{r^{n+1}})\, P_n(\cos\theta)$$

where $P_n(\cos\theta)$ is the Legendre polynomials and

$$\int_{-\pi}^{\pi} P_m(\cos\theta)\, P_n(\cos\theta)\, d(\cos\theta) = \frac{2}{2n+1}\,\delta_{mn}$$

$P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, $P_2(\cos\theta) = (3\cos^2\theta\text{-}1)/2$, $P_3(\cos\theta) = (5\cos^3\theta\text{-}3\cos\theta)/2$,

2. General solution of $\frac{\partial}{\partial\rho}(\rho\frac{\partial V}{\partial\rho}) + \frac{1}{\rho}\frac{\partial^2 V}{\partial\phi^2} = 0$ in cylindrical coordinates

$$V(\rho,\phi) = D_0 + A_0\, ln\rho + \sum_{\nu}(A_\nu \rho^\nu + \frac{B_\nu}{\rho^\nu})\,\{C_\nu \sin\nu\phi + D_\nu \cos\nu\phi\}$$

For unrestricted ϕ, ν = positive non-zero integer.

$$\int_{-\pi}^{\pi}\sin m\phi\, \sin n\phi\, d\phi = \pi\,\delta_{mn},\quad \int_{-\pi}^{\pi}\cos m\phi\, \cos n\phi\, d\phi = \pi\,\delta_{mn},\quad \int_{-\pi}^{\pi}\sin m\phi\, \cos n\phi\, d\phi = 0$$

3. General solution of $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$, $\quad V(x,y) = \sum_{a_n}\{\exp(\pm a_n x)\}\,\{\exp(\pm i a_n y)\}$

$e^{\pm ix} = \cos x \pm i\sin x$, $\quad e^{\pm x} = \cosh x \pm \sinh x$

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}dx = \frac{a}{2}\delta_{mn},\quad \int_0^a \cos\frac{m\pi x}{a}\cos\frac{n\pi x}{a}dx = \frac{a}{2}\delta_{mn},\quad \int_0^a \sin\frac{m\pi x}{a}\cos\frac{n\pi x}{a}dx = 0,$$

VECTOR DERIVATIVES

CARTESIAN. $d\mathbf{l} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$; $d\tau = dx\,dy\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial x}\hat{\imath} + \frac{\partial t}{\partial y}\hat{\jmath} + \frac{\partial t}{\partial z}\hat{k}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k}$$

Laplacian. $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

SPHERICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl. $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

Laplacian. $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

CYLINDRICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z}$; $d\tau = r\,dr\,d\phi\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial\phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla \times \mathbf{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi}$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial\phi}\right]\hat{z}$$

Laplacian. $$\nabla^2 t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

TRIPLE PRODUCTS

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

PRODUCT RULES

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

SECOND DERIVATIVES

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_a^b (\nabla f) \cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\int_{\text{volume}} (\nabla \cdot \mathbf{A})\, d\tau = \oint_{\text{surface}} \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int_{\text{surface}} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_{\text{line}} \mathbf{A} \cdot d\mathbf{l}$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}$ coul^2/N-m^2	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}$ N/A^2	(permeability of free space)
$c = 3.00 \times 10^8$ m/sec	(speed of light)
$e = 1.60 \times 10^{-19}$ coul	(charge of the electron)
$m = 9.11 \times 10^{-31}$ kg	(mass of the electron)

CONVERSION FROM SPHERICAL TO CARTESIAN COORDINATES

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{i} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} \\ \hat{j} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi} \\ \hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k} \\ \hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k} \\ \hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} \end{cases}$$