

Graduate diagnostic test, section II
Theoretical Mechanics and Statistical Physics & Thermodynamics
August 24, 2012, 2 pm-5 pm

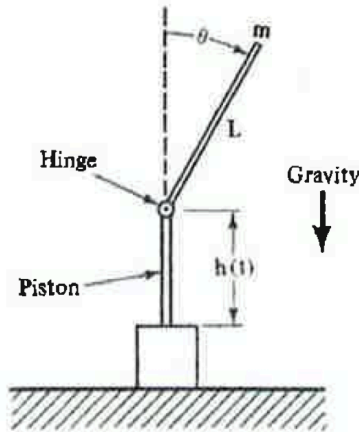
This is a three-hour closed book diagnostic test. Please show us what you can do with the assistance only of the attached formula sheet. Ask your proctor for clarification if the text is unclear. Provide detailed reasoning using text, sketches and equations and express your answers using variables defined in the problem. Describe your ideas for partial credit even if you cannot solve the problem.

Problem 7

Find the Lagrangian and the equation of motion of the following system: A point mass, m , on a massless rod of length L . The rod is hinged (frictionless hinge). The hinge is constrained to oscillate vertically with harmonic motion:

$$h(t) = h_0 \cos \omega t .$$

The only degree of freedom is θ , which measures the angle of the rod from the vertical. You do not have to solve the equation of motion.



Problem 8

Consider a particle moving in a central field of force with potential being given by:

$$V(r) = \frac{1}{2}kr^2$$

If we consider the radial motion only:

- (a) What is the “effective” potential in which the radial motion occurs? Sketch and label the contributions.
- (b) What is the condition for circular motion in terms of “effective” potential?
- (c) Calculate the angular frequency for circular orbits.

Problem 9

A particle is constrained to move on the surface of a sphere of radius, L , in a constant gravitational force field ($F_x=F_y=0$, $F_z=-mg$). Set up the Hamiltonian formulation of the problem in terms of two angles defining the positions of a point on the sphere. Give at least two constants of the motion. You do not need to solve Hamilton's Equations.

Problem 10

- Describe the Carnot process and derive its efficiency for use as a heat engine.
- For a three-dimensional solid with N atoms, what is the maximum number of phonon modes?
- For a cube of volume $V = L^3$, what is the maximum number of electromagnetic modes?

Problem 11

A system at temperature T includes two particular states, **A** and **B**. The energy of state **A** is ε_A and the energy of state **B** is $\varepsilon_B = \varepsilon_A + \delta\varepsilon$. What is the relative probability of the system being in state **A** relative to being in state **B**?

Problem 12

Consider a system of $N \gg 1$ non-interacting particles in which the energy of each particle can have two (and only two) distinct values: 0 and E ($E > 0$). The occupation numbers of the zero and E energy levels are n_0 and n_1 respectively. The fixed total energy of the system is U .

- Find the entropy of the system.
- Find the temperature as a function of U .
- For what range of values of n_0 is the temperature, $T < 0$?

Formulae

Avogadro's number: $N_A = 6.02 \times 10^{23} \text{ mole}^{-1}$

Ideal gas constant: $R = 8.3 \text{ J K}^{-1} \text{ mol}^{-1}$

Boltzmann constant: $k_B = 1.38 \times 10^{-23} \text{ J K}^{-1}$

$$R = N_A k_B$$

One mole of an ideal gas occupies 22.4 liters at standard temperature and pressure.

Electron mass: $m_e = 9.11 \times 10^{-28} \text{ g}$

Bose-Einstein and Fermi-Dirac distribution functions:

$$N_b = \frac{1}{e^{\frac{\epsilon - \mu}{k_B T}} - 1} \quad N_f = \frac{1}{e^{\frac{\epsilon - \mu}{k_B T}} + 1}$$

where ϵ is the energy and μ is the chemical potential.

First Law of Thermodynamics:

$$dQ = dU + dW$$

Thermodynamic identity

$$\tau d\sigma = dU + p dV$$

τ is $k_B T$.

Helmholtz Free Energy

$$F = U - \tau \sigma$$

$$F = -k_B T \ln Z \quad \mu = \left(\frac{\partial F}{\partial N} \right)_{T, V}$$

where F is the free energy, Z is the partition function, μ is the chemical potential, and N is the number of particles.

$$U = \tau^2 \frac{\partial \ln Z}{\partial \tau}$$

where U is the average system energy

$$\frac{1}{k_B} \left(\frac{\partial F}{\partial T} \right)_V = -\sigma \quad \left(\frac{\partial F}{\partial V} \right)_T = -p$$

where σ is the entropy and p is the pressure.

$$\Delta S = \int_1^2 \frac{dQ}{T}$$

where ΔS is the entropy change from state 1 \rightarrow 2, Q is the heat, and T is temperature.

$$W_{1 \rightarrow 2} = \int_1^2 p dV \quad pV = nRT$$

where $W_{1 \rightarrow 2}$ is the work done on a path from 1 \rightarrow 2, p is pressure, V is volume, n is the number of moles, R is the ideal gas constant and T is temperature.