Electromagnetism

2. A large (infinite) parallel-plate capacitor is filled with a positively charged dielectric of uniform volume charge density $\rho$ and dielectric constant $\varepsilon$. The plate at $+d/2$ is grounded and the plate at $-d/2$ is held at potential $V$.

![Diagram of a parallel-plate capacitor]

(a) [15 pts] Calculate the electric field in the gap as a function of $z$, the distance from the gap center and applied potential. At what potential $V_0$ does the electric field $E_z$ vanish at $z = -d/2$?

(b) [5 pts] What are the magnitudes and signs of the surface charge densities on the plates when the capacitor is biased at $V_0$?

4. A circuit containing a capacitor of capacitance $C$, a resistor of resistance $R$, and an inductor of inductance $L$ in parallel as shown below. The circuit is prepared by charging the capacitor to charge $Q_0$. At time $t = 0$, the switch is closed allowing current to begin to flow ($I(0) = 0$).

![Diagram of a parallel circuit]

(a) [15 pts] Calculate the voltage $V$ and current $I$ across the capacitor as functions of time.

(b) [5 pts] For what range of resistances is the solution oscillatory? What is the form of the solution at the transition from oscillatory to non-oscillatory (critically damped)?
2. (25 points) A grounded spherical conductor of radius \(a\) is centered inside an insulating spherical shell of radius \(b\). A surface charge density is placed on the insulating shell such that its potential is \(V(\theta) = V_0 P_2(\cos \theta)\), where \(P_2\) is the second Legendre polynomial. The potential vanishes far from the shell. Determine the angular dependence of the surface charge density \(\sigma(\theta)\) on the insulating shell. **Hint:** You will need to find the potential both inside and outside the insulating shell.

![Diagram](image)

**Figure 1: Problem 4**

4. (20 points) Two long vertical parallel wires are connected at their top by a third wire and by a bar of mass \(m\) and resistance \(R\) (see Figure 1). The bar is attached to the wires, but free to slide, without friction, vertically along the wires. A uniform, constant magnetic field \(\vec{B}\) is pointing into the page. Assuming the wire has negligible resistance and is fixed in space on the surface of the Earth:

(a) What is the velocity of the bar as a function of time, \(t\) (assuming the bar starts at rest).

(b) What is the energy expended in heating the resistor (Joule heating) per unit time in the limit \(t \to \infty\)?

1. (20 points) Consider two conducting spheres of radius \(a\) and \(b\) \((a < b)\). The region between the spheres is filled with an inhomogeneous, linear dielectric with \(\varepsilon = \varepsilon_0/(c - \alpha r)\) where \(c\) and \(\alpha\) are constants. A charge \(q\) is placed on the inner sphere.

(a) Find the electric displacement vector \(\vec{D}\) in the region between the spheres.

(b) Find the volume polarization charge density, \(\rho_P\), in the dielectric.

(c) Show **explicitly** that the total polarization charge is zero.
4. (15 points) A plane electromagnetic wave traveling in a vacuum is given by

\[ \mathbf{E} = E_0\exp[i(kz - \omega t)] \hat{j} \]

where \( E_0 \) is real. A circular loop of radius \( a \), \( N \) turns, and resistance \( R \) is located with its center at the origin. The loop is oriented so that a diameter lies along the \( z \)-axis and the plane of the loop makes an angle \( \theta \) with the \( y \)-axis.

(a) Find the emf induced in the loop.

(b) For part (a) we must assume that \( a \ll \lambda \). Why?

5. (15 points) A cylindrical conductor of radius \( b \) contains a cylindrical hole of radius \( a \); the axis of the hole is parallel to the axis of the conductor and a distance \( s \) away from it, as shown in the diagram. A current \( I \) flows in the conductor.

(a) Find the \( \mathbf{B} \)-field in the hole on the diameter that coincides with a diameter of the conductor.

(b) Find the \( \mathbf{B} \)-field at a point outside the conductor at a distance \( d \) from the axis and on the same diameter as in part (a).

3 A long coaxial cable carries current \( I \). The current flows down the surface of the inner cylinder with radius \( a \) and then back along the outer cylinder with radius \( b \) (see below). Find the magnetic energy stored in a section of length \( l \). What is the self-inductance \( L \) of this section of the cable?
5. Analyze the process of Compton scattering. A photon with initial energy $E_0$ is scattered by an electron that is initially at rest. Find the energy $E$ of the outgoing photon as a function of the scattering angle $\theta$ (see below). Now rewrite your answer to solve for the wavelength of the outgoing photon ($\lambda$) as a function of its initial wavelength ($\lambda_0$) and the Compton wavelength of the electron ($\lambda_c = h/m_e c$)

2. (a) What is the lowest non-vanishing multipole of a cube with alternating charges of equal magnitude on the 8 corners?

(b) A single point charge, $q$, moves along the $y$-axis according to $y(t) = \sin(\omega t)$. What is the power radiated? What is corresponding multipole of the radiation?

(c) A perfectly conducting sphere of total charge $Q$ undergoes a very small radial oscillation such that the radius of the sphere is given by $R + r_0 \sin(\omega t)$ where $r_0 \ll R$. Which multipole components contribute to the overall radiation? What is the total power radiated?

5. Consider a ball of radius $a$ that has charge $Q$ spread evenly on its surface. The ball is rotating on its axis with angular frequency $\omega$.
   a) Find a vector expression for the current density as a function of $\theta$.
   b) Find the vector potential far away from the ball. Ignore terms smaller than $a/R$, where $R$ is the distance to the observation point.
   c) Using the vector potential, calculate the magnetic field.
   d) From the form of your answer, find the magnetic dipole moment of the spinning ball.
1. (a) Consider two conducting hemispherical shells placed concentrically to form a full sphere with a tiny insulating gap between the hemispheres. One hemisphere is grounded, the other is held at the potential \(q_0\). What is the potential at the center of the sphere? Why?

(b) Two insulated, identically charged spheres, suspended from strings are in static equilibrium.

A very large grounded, conducting plate is now placed underneath the two spheres. After a new static equilibrium is achieved, are the spheres closer together, further apart, or unaffected? Why?

**Problem 2:** Two semi-infinite grounded metal planes lie parallel to the \(xz\) plane, one at \(y = 0\), the other at \(y = a\). The planes are infinite in the \(z\) direction, but semi-infinite in the \(x\) direction. The two semi-infinite planes are connected at their boundary at \(x = 0\) with an infinite strip that is insulated from the two plates, but maintained at a specified potential \(V_0\). See the figure below. By applying appropriate boundary conditions and solving Laplace’s equation find the potential inside this space between the plates for \(x > 0\).

**Problem 5:** A plane, monochromatic electromagnetic wave propagates in an infinite medium of conductivity \(\sigma\). The medium has negligible magnetic or electric polarizability. Starting with Maxwell’s equations and Ohm’s Law

(a) Derive a wave equation obeyed by the electric field.

(b) Derive a dispersion relation relating the wavenumber \(k\) to the angular frequency \(\omega\).

(c) Show that the amplitude of the wave decreases exponentially in the direction of propagation, and obtain an expression for the attenuation length scale in the limit of poor conductivity (defined as \(\sigma \ll \omega \varepsilon_0\)).
Useful Equations

- Maxwell’s Equations:
  \[ \nabla \cdot \mathbf{D} = \rho(x) \quad \nabla \cdot \mathbf{B} = 0 \quad \partial \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \partial \mathbf{H} = \mathbf{j} \quad \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \quad \nabla \times \mathbf{H} = \mathbf{j} + \partial \mathbf{D}/\partial t \]

- Boundary Conditions: \( \hat{n} \) is normal to the plane boundary, \( \sigma \) and \( \mathbf{K} \) are surface charge and current densities.
  \[ \hat{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \sigma \quad \hat{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0 \]
  \[ \hat{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \quad \hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{K} \]

- Static Potentials: \( \varphi \) and \( \mathbf{A} \) are the electric scalar and magnetic vector potentials,
  \[ \mathbf{E} = -\nabla \varphi \quad \mathbf{H} = \nabla \times \mathbf{A} \]

- Potential of a dipole:
  \[ \varphi(\mathbf{r}) = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi \varepsilon_0 r^3} \]

- Dipole Moment:
  \[ \mathbf{p} = \int \mathbf{x} \rho(x) d^3x \]

- Lorentz Force:
  \[ \frac{d\mathbf{p}}{dt} = q \mathbf{E} + q \mathbf{v} \times \mathbf{B} \]

- Homogeneous Media:
  \[ \mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{F} \]
  \[ \mathbf{B} = \mu \mathbf{H} = \mu_0 (\mathbf{H} + \mathbf{M}) \]

- Definitions:
  - Capacitance - \( C = Q/V \)
  - Resistance - \( V = IR \)
  - Inductance - \( V = -LdI/dt \)
Electric dipole

\[ V_{\text{dipole}} = \frac{1}{4\pi \varepsilon_0} \frac{P \cdot r}{r^3} \]

\[ E_{\text{dipole}} = \frac{1}{4\pi \varepsilon_0} \frac{P}{r^3} [2\cos \theta \hat{\ell}_1 + \sin \theta \hat{\theta}_1] \]

For a dielectric with polarization \( P \), the potential is

\[ V(r) = \frac{1}{4\pi \varepsilon_0} \int \frac{P(r') \cdot (r - r')}{|r - r'|^3} \, dr' \]

\[ = \frac{1}{4\pi \varepsilon_0} \int \frac{-\nabla' \cdot P(r')}{|r - r'|} \, dr' + \frac{1}{4\pi \varepsilon_0} \int \frac{P(r') \cdot \hat{n}_1}{|r - r'|} \, da' \]

\[ D = \varepsilon_0 E + P \]

Quadrupole moment tensor:

\[ Q_{ij} = \int (3x_i x_j - r^2 \delta_{ij}) \rho(r) \, dr \]

Current density:

\[ J = n q v = \rho \, v \]

Ohm's law:

\[ J = \sigma E \]

Capacitor:

\[ Q = CV \]

Continuity Equation:

\[ \nabla \cdot J + \frac{\partial \rho}{\partial t} = 0, \]

Biot-Savart Law

\[ B(r) = \frac{\mu_0}{4\pi} \int \frac{l \, d' \times (r - r')}{|r - r'|^3} = \frac{\mu_0}{4\pi} \int \frac{J(r') \times (r - r')}{|r - r'|^3} \, dr' \]

Vector potential

\[ A(r) = \frac{\mu_0}{4\pi} \int \frac{J(r')}{|r - r'|} \, dr' \]

\[ dA(r) = \frac{\mu_0}{4\pi} \frac{df}{r} \]

\[ \vec{E} = -\frac{\partial A}{\partial t} - \nabla V \]
Some useful mathematical information:

1. General solution of \( \frac{\partial}{\partial x}(x^2 \frac{\partial V}{\partial x}) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}(\sin \theta \frac{\partial V}{\partial \theta}) = 0 \) in spherical coordinates

\[
V(r, \theta) = \sum_{n=0}^{\infty} \left( A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \theta)
\]

where \( P_n(\cos \theta) \) is the Legendre polynomials and

\[
\int_{-\pi}^{\pi} P_m(\cos \theta) P_n(\cos \theta) \, d(\cos \theta) = \frac{2}{2m+1} \delta_{mn}
\]

\( P_0(\cos \theta) = 1, \ P_1(\cos \theta) = \cos \theta, \ P_2(\cos \theta) = (3\cos^2 \theta - 1)/2, \ P_3(\cos \theta) = (5\cos^3 \theta - 3\cos \theta)/2, \)

2. General solution of \( \frac{\partial}{\partial \rho}(\rho \frac{\partial V}{\partial \rho}) + \frac{1}{\rho} \frac{\partial^2 V}{\partial \phi^2} = 0 \) in cylindrical coordinates

\[
V(\rho, \phi) = D_0 + A_0 \ln \rho + \sum_{n} \left( A_n \rho^n + \frac{B_n}{\rho^n} \right) \left( C_n \sin n \phi + D_n \cos n \phi \right)
\]

For unrestricted \( \phi, \ \nu = \text{positive non-zero integer} \).

\[
\int_{-\pi}^{\pi} \sin \nu \phi \sin \phi \, d\phi = \pi \delta_{\nu \nu}, \quad \int_{-\pi}^{\pi} \cos \nu \phi \cos \phi \, d\phi = \pi \delta_{\nu \nu}, \quad \int_{-\pi}^{\pi} \sin \nu \phi \cos \phi \, d\phi = 0
\]

3. General solution of \( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \), \( V(x,y) = \sum_{n} \{ \exp(\pm \pi \nu x) \} \{ \exp(\pm \pi \nu y) \} \)

\( e^{\pm ix} = \cos x \pm i \sin x, \quad e^{\pm ix} = \cosh x \pm i \sinh x \).

\[
\int_{0}^{\pi} \sin a \sin bx \, dx = \frac{a}{2} \delta_{mn}, \quad \int_{0}^{\pi} \cos a \cos bx \, dx = \frac{a}{2} \delta_{mn}, \quad \int_{0}^{\pi} \sin a \cos bx \, dx = 0
\]
VECTOR DERIVATIVES

CARTESIAN. \( dl = dx \hat{i} - dy \hat{j} - dz \hat{k} \); \( d\tau = dx \, dy \, dz \)

Gradient. \( \nabla l = \frac{\partial l}{\partial x} \hat{i} - \frac{\partial l}{\partial y} \hat{j} - \frac{\partial l}{\partial z} \hat{k} \)

Divergence. \( \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \)

Curl. \( \nabla \times \mathbf{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{i} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{j} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{k} \)

Laplacian. \( \nabla^2 l = \frac{\partial^2 l}{\partial x^2} + \frac{\partial^2 l}{\partial y^2} + \frac{\partial^2 l}{\partial z^2} \)

SPHERICAL. \( dl = dr \hat{r} + r \, d\theta \hat{\theta} + r \sin \theta \, d\phi \hat{\phi} \); \( d\tau = r^2 \sin \theta \, dr \, d\theta \, d\phi \)

Gradient. \( \nabla l = \frac{\partial l}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial l}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial l}{\partial \phi} \hat{\phi} \)

Divergence. \( \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \)

Curl. \( \nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{r} \)
\( + \frac{1}{r} \left[ \frac{\sin \theta}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial v_\phi}{\partial r} \right] \hat{\theta} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (rv_\phi) - \frac{\partial v_r}{\partial \theta} \right] \hat{\phi} \)

Laplacian. \( \nabla^2 l = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial l}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial l}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 l}{\partial \phi^2} \)

CYLINDRICAL. \( dl = dr \hat{r} + r \, d\phi \hat{\phi} + dz \hat{z} \); \( d\tau = r \, dr \, d\phi \, dz \)

Gradient. \( \nabla l = \frac{\partial l}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial l}{\partial \phi} \hat{\phi} + \frac{\partial l}{\partial z} \hat{z} \)

Divergence. \( \nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \)

Curl. \( \nabla \times \mathbf{v} = \left[ \frac{1}{r} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{r} + \left[ \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right] \hat{\phi} \)
\( + \frac{1}{r} \left[ \frac{\partial}{\partial r} (rv_\phi) - \frac{\partial v_r}{\partial \theta} \right] \hat{z} \)

Laplacian. \( \nabla^2 l = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial l}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 l}{\partial \phi^2} + \frac{\partial^2 l}{\partial z^2} \)
VECTOR IDENTITIES

TRIPLE PRODUCTS

(1) \( A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B) \)
(2) \( A \times (B \times C) = B(A \cdot C) - C(A \cdot B) \)

PRODUCT RULES

(3) \( \nabla(fg) = f(\nabla g) + g(\nabla f) \)
(4) \( \nabla(A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla)B + (B \cdot \nabla)A \)
(5) \( \nabla \cdot (fA) = f(\nabla \cdot A) + A \cdot (\nabla f) \)
(6) \( \nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B) \)
(7) \( \nabla \times (fA) = f(\nabla \times A) - A \times (\nabla f) \)
(8) \( \nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B + A(\nabla \cdot B) - B(\nabla \cdot A) \)

SECOND DERIVATIVES

(9) \( \nabla \cdot (\nabla \times A) = 0 \)
(10) \( \nabla \times (\nabla f) = 0 \)
(11) \( \nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A \)

FUNDAMENTAL THEOREMS

Gradient Theorem: \( \int_a^b (\nabla f) \cdot dl = f(b) - f(a) \)

Divergence Theorem: \( \int_{\text{volume}} (\nabla \cdot A) \, d\tau = \int_{\text{surface}} A \cdot d\mathbf{a} \)

Curl Theorem: \( \int_{\text{surface}} (\nabla \times A) \cdot d\mathbf{a} = \int_{\text{lines}} A \cdot dl \)
FUNDAMENTAL CONSTANTS

\( \varepsilon_0 = 8.85 \times 10^{-12} \text{ coul}^2/\text{N-m}^2 \)  
(permittivity of free space)

\( \mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2 \)  
(permeability of free space)

\( c = 3.00 \times 10^8 \text{ m/sec} \)  
(speed of light)

\( e = 1.60 \times 10^{-19} \text{ coul} \)  
(charge of the electron)

\( m = 9.11 \times 10^{-31} \text{ kg} \)  
(mass of the electron)

CONVERSION FROM SPHERICAL TO CARTESIAN COORDINATES

\[
\begin{align*}
  x &= r \sin \theta \cos \phi \\
  y &= r \sin \theta \sin \phi \\
  z &= r \cos \theta
\end{align*}
\]

\[
\begin{align*}
  \hat{i} &= \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi} \\
  \hat{j} &= \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi} \\
  \hat{k} &= \cos \theta \hat{r} - \sin \theta \hat{\theta}
\end{align*}
\]

\[
\begin{align*}
  r &= \sqrt{x^2 + y^2 + z^2} \\
  \theta &= \tan^{-1}(\sqrt{x^2 + y^2}/z) \\
  \phi &= \tan^{-1}(y/x)
\end{align*}
\]

\[
\begin{align*}
  \hat{r} &= \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k} \\
  \hat{\theta} &= \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k} \\
  \hat{\phi} &= -\sin \phi \hat{i} + \cos \phi \hat{j}
\end{align*}
\]
Quantum Mechanics

Problem 1 (15 points)
Consider a non-relativistic particle with mass $m$ subject to the following one dimensional potential; $g$ being a positive constant:

$$V(x) = -g(\delta(x+a) + \delta(x) + \delta(x-a))$$

(a) Duly considering symmetry, write down the functional form of the bound state wave functions and sketch these. (5 points)

(b) Argue that one of these states unbinds as $a \to 0$. (4 points)

(c) Determine the smallest value of $a$, which supports more than one bound state. (6 points)

Problem 4 (15 points)
In this problem we shall consider a non-relativistic particle of mass $m$ in the infinite square well potential well shown in Fig. 1.

(a) Determine the energy levels and normalized wave functions for $V_0 = 0$. (5 points)

(b) What condition must $V_0$ satisfy in order that perturbation theory can be used to determine its effects on the ground state? (4 points)

(c) Determine the shift in the ground state energy to lowest order in $V_0$. (6 points)

Fig. 1 Sketch of the potential analyzed in problem 4.
Problem 5

Consider two spin 1/2 particles whose Hamiltonian is given by:

\[ H_0 = K s_1 s_2. \]  

(7)

- Find energy levels and the corresponding wave functions;
- Suppose that magnetic field of strength \( H \) is switched on. Assume that magnetic moments of the two particles have equal absolute values \( \mu \) but different signs. Find energy levels.

Problem 6 (15 points)

A non-relativistic particle with mass \( m \) and momentum \( p = \hbar k \) scatters from a spherical potential well:

\[ V(r) = \begin{cases} 
-V_0, & r < a \\
0, & r > a
\end{cases} \]

(c) Compute the differential cross section in the first Born approximation (10 points)

(d) Determine the total scattering cross section in the limit \( k a \to 0 \) (5 points)

1. (10 pts) A particle with mass \( m \) is in an infinite square well of width \( a \) centered at \( x = 0 \). The particle has as its initial wave function an even mixture of the first two normalized stationary states with energies \( E_1 \) and \( E_2 \) respectively:

\[ \psi(x, 0) = A[\psi_1(x) + \psi_2(x)]. \]

A) What are the energies \( E_1 \) and \( E_2 \)?

B) Normalize \( \psi(x, 0) \).

C) Find \( \psi(x, t) \) and \( |\psi(x, t)|^2 \). Express \( |\psi(x,t)|^2 \) in terms of sinusoidal functions of time.

D) Compute \( \langle x(t) \rangle \). Notice that it oscillates. What are the frequency and amplitude of the oscillation?

3. (20 pts) Two noninteracting particles are in an infinite one dimensional square well, one in state \( \psi_m \) and the other in state \( \psi_n \). Calculate \( \langle (x_1 - x_2)^2 \rangle \) for the three cases of the particles being 1) distinguishable, 2) identical bosons and 3) identical fermions.
1. Consider a particle of spin $\hbar/2$. Assume the particle is at a fixed position so it has no other degrees of freedom. The Hamiltonian is given by:

$$\hat{H} = \mu B \hat{S}_x$$

where $\mu B$ is positive and $\hat{S}_x$ is the $x$-component of the spin operator. At time $t = 0$ the particle is in an eigenstate of $\hat{S}_x$ with eigenvalue $\hbar/2$. What is the probability that a measurement of $S_x$ will yield a value $\hbar/2$ at time $t > 0$?

**Problem 3 (25 points)**

A charged harmonic oscillator with in a uniform electric field $\mathcal{E}$ has the following Hamiltonian operator (position basis):

$$\hat{H}_\mathcal{E} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2 - q \mathcal{E} x$$

(a) By re-writing the potential energy term in a quadratic form show that the eigenvalues and eigenstates can be written as follows

$$E_n = \hbar \omega (\frac{1}{2} + n) - \frac{q^2 \mathcal{E}^2}{2m \omega^2}$$

$$|n\rangle_\mathcal{E} = \hat{T}(d) |n\rangle_{\mathcal{E}=0}$$

Here $\hat{T}(d) = \exp(-ip/\hbar)$ is the translation operator and $d = \frac{q \mathcal{E}}{m \omega^2}$ (5 points)

Now assume that the particle is in the ground state $|0\rangle_\mathcal{E}$ when the electric field is abruptly removed. Denote by $P_n$ the probability that a measurement of the energy of the system following that treatment will yield the value $\hbar \omega (\frac{1}{2} + n)$.

(b) In the following you will be guided towards and analytical expression for $P_n$. To start off, derive an expression for $P_n$ that involves $\hat{T}(d)$. (4 points)

(c) Show that $\exp(\lambda \hat{a}) |0\rangle = |0\rangle$. Here $\hat{a}$ is the lowering operator for the harmonic oscillator. (4 points)

(d) Show that $\langle n | \exp(\lambda \hat{a}^*) |0\rangle = \frac{\lambda^n}{\sqrt{n!}}$. Here $\hat{a}^*$ is the raising operator for the harmonic oscillator. (5 points)

(e) It can be shown that $\exp(\hat{A}) \exp(\hat{B}) = \exp(\hat{A} + \hat{B}) \exp(\frac{1}{2} [\hat{A}, \hat{B}])$ when both operators $\hat{A}$ and $\hat{B}$ commute with $[\hat{A}, \hat{B}]$. Use this result and the results from (b)-(d) to show that

$$P_n = \exp(- \lambda^2) \frac{1}{n!} \lambda^n$$

where $\lambda = d \sqrt{\frac{m \omega}{2 \hbar}}$ (7 points)
2. The Hamiltonian and operators $\hat{A}$ and $\hat{B}$ for a certain three level system are represented by the following matrices where $\omega$, $\lambda$ and $\mu$ are positive real numbers:

$$\hat{H} \rightarrow \hbar \omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \hat{A} \rightarrow \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \hat{B} \rightarrow \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

A) (5 pts) Determine whether $\hat{A}$ and $\hat{B}$ are physical observables

B) (5 pts) Find the simultaneous eigenstates of $\hat{H}$ and $\hat{A}$ if they exist.

C) (5 pts) Find an example of states for which $\langle A \rangle$ is time independent if such states exist.

D) (5 pts) Establish a lower bound on the products of the uncertainties $\Delta A \Delta B$ for measurements in the quantum state described by

$$|\psi> \rightarrow \frac{1}{\sqrt{3}} \begin{pmatrix} i \\ 1 \\ -i \end{pmatrix}.$$

4. An electron in a hydrogen atom is in the $n=2$ state. A constant electric field, $\vec{E} = \xi \hat{z}$, in the $\hat{z}$ direction is applied to the atom. Ignore spin in this problem.

A) (4 pts) Find the first order corrections to the energy levels.

B) (8 pts) Find the second order corrections to the energy levels.

C) (8 pts) Find the new wave functions for the atom after second order corrections.

5. Suppose you were to try to find the ground state of the hydrogen atom by guessing that it was a gaussian wave function like

$$\psi(r, \theta, \phi) = Ae^{-br^2}.$$

A) (4 pts) Find $A$ in terms of $b$.

B) (8 pts) For a given value of $b$ find the expectation value of the Hamiltonian.

C) (8 pts) Use the variational method to find the lowest bound state energy for this form of the wave function.

**Problem 5 (20 points)**

A deuteron is a bound state of a proton and a neutron with separation $r = r_p - r_n$. It can be well approximated by a 3-dimensional potential well

$$V(r) = -V_0, \quad r \leq R; \quad V(r) = 0, \quad r > R$$

Find the solution for the radial part of the Schrödinger equation $u(r) = rR(r)$ assuming there exists a bound state with $E = -[E_0]$ in the angular momentum state $l = 0$. (You may assume both $p$ and $n$ have mass $m$.) Match the boundary conditions to obtain an equation relating $V_0$ and $E_0$ to $R$ and $m$. (It will involve trigonometric functions.) What can you say about the minimum depth $V_{0 \text{min}}$ for a bound state to exist?
1. [10 pts] A particle is confined in a one-dimensional potential well having the form \( V(x) = \alpha x^4 \). Using the uncertainty principle, estimate the ground state energy of the particle.

2. [10 pts] A 1-dimensional particle of mass \( m \) is constrained to slide (freely) in a circular 1-D path of radius \( R \).

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{circle.png}
\end{array}
\]

What are the wavefunction \( \psi(\phi) \) and allowed energies of the particle?

3. A 1-dimensional particle of mass \( m \) and positive energy \( E \) is incident from the left \( (x < 0) \) on a delta-function potential barrier located at the origin of the coordinate system, \( V(x) = V_0 \delta(x) \).

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{delta.png}
\end{array}
\]

(a) [10 pts] What boundary conditions apply to the wavefunction \( \psi(x) \) at the barrier \( (x = 0) \)?

(b) [10 pts] What fraction of a population of incident particles of energy \( E \) are transmitted through the barrier into the region \( x > 0 \)?

**Problem 3**—If \( \hat{A} \) and \( \hat{B} \) are arbitrary, possibly explicitly time-dependent operators, show that

\[
\frac{d}{dt} \langle \hat{A} \hat{B} \rangle = \langle \frac{\partial \hat{A}}{\partial t} \hat{B} \rangle + \langle \hat{A} \frac{\partial \hat{B}}{\partial t} \rangle + \frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \hat{B} \rangle + \frac{i}{\hbar} \langle \hat{A} [\hat{H}, \hat{B}] \rangle ,
\]

where \( \hat{H} \) is the Hamiltonian and \( \langle \cdots \rangle \) denotes expectation value of an operator in some quantum mechanical state.
FORMULÆ

Normalization (one dimensional systems)

\[ <\psi|\psi> = \int_{-\infty}^{\infty} dx \ <\psi|x><x|\psi> = \int_{-\infty}^{\infty} dx \ \psi^*(x)\psi(x) = 1. \]

Expectation values (one dimensional systems)

\[ <\hat{x}> = \int_{-\infty}^{\infty} dx \ <\psi|x><x|\psi> = \int_{-\infty}^{\infty} dp \ <\psi|p><p|\psi> = \int_{-\infty}^{\infty} x \ \psi^*(x)\psi(x)dx = \int_{-\infty}^{\infty} x |\psi(x)|^2 dx, \]

\[ <\hat{x}^2> = \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx = \int_{-\infty}^{\infty} dx \ <\psi|x^2> = \int_{-\infty}^{\infty} <x|\psi> \ , \text{ etc.} \]

The general form of the Heisenberg uncertainly relation, for arbitrary hermitean operators \( \hat{A} \) and \( \hat{B} \) is

\[ \Delta \hat{A}\Delta \hat{B} \geq \frac{1}{2} |<\psi|[\hat{A},\hat{B}]|\psi>|, \]

where

\[ (\Delta \hat{A})^2 \equiv <\psi|(\hat{A}^2 - <\psi|\hat{A}|\psi>)|\psi>, <\psi|\psi> = 1, \]

etc. for \( (\Delta \hat{B})^2 \)

Time dependent Schrödinger equation, Heisenberg and Schrödinger pictures.

In all the following formulæ, \( \hat{H} \) stands for the Hamiltonian of the system. The subscripts \( S \) and \( H \) refer to the Schrödinger and Heisenberg pictures, respectively.

i) Schrödinger picture:

\[ i\hbar \frac{\partial |\psi_S>}{\partial t} = \hat{H}|\psi_S>. \]

The operators, \( \hat{\Omega} \), have only explicit time dependence.

ii) Heisenberg picture:

If \( \hat{\Omega} \) is any operator, its equation of motion reads:

\[ \frac{d\hat{\Omega}_H}{dt} = \frac{i}{\hbar} [\hat{H},\hat{\Omega}_H] + \frac{\partial \hat{\Omega}_H}{\partial t}. \]
If $\hat{H}$ is not explicitly time dependent the last term on the right vanishes. The state vectors are time independent.

iii) Relationship between the Heisenberg and Schrödinger pictures. All formulae are written in such a way that the Schrödinger and Heisenberg pictures coincide at $t = 0$.

If the Hamiltonian does not depend explicitly on time, it is the same in both pictures:

$$\hat{H}_H = \hat{H}_S.$$ 

For any other operator:

$$\hat{\Omega}_H = \exp(\frac{i}{\hbar} \hat{H} t) \hat{\Omega}_S \exp(-\frac{i}{\hbar} \hat{H} t).$$

Relationship between state vectors:

$$|\psi_S \rangle = \exp(-\frac{i}{\hbar} \hat{H} t) |\psi_H \rangle.$$ 

Canonical commutation relations.

$$[\hat{z}_i, \hat{p}_j] = i\hbar \delta_{ij}.$$ 

In the coordinate basis,

$$\hat{z}_i \rightarrow z_i \text{ (multiplication by } z_i) \hat{p}_i \rightarrow \frac{\hbar}{i} \frac{\partial}{\partial z_i}.$$ 

Angular momentum.

In what follows, all angular momenta are measured in units of $\hbar$, hence they are dimensionless.

The commutation relation for angular momentum is

$$[\hat{j}_i, \hat{j}_j] = i\epsilon_{ijk} \hat{j}_k,$$

where $\epsilon_{ijk}$ is the Levi-Civita tensor density, $[\epsilon_{ijk} = 0$ for any two or more indices equal, $\epsilon_{ijk} = 1$ for cyclic and $\epsilon_{ijk} = -1$ for anticyclic permutations of $1, 2, 3$].

The step-up (- down) angular momentum operators $\hat{j}_+ (\hat{j}_-)$ are defined as

$$\hat{j}_+ (-) \equiv \hat{j}_1 \quad \hat{j}_2 \equiv \hat{j}_{x+(-)} \hat{j}_y$$

Orbital angular momentum:

$$\hat{L} = \frac{1}{\hbar} \hat{z} \times \hat{p}, \hat{L}_z = \frac{1}{i} \frac{\partial}{\partial \varphi}.$$
Spin angular momentum, for spin 1/2:

\[ \hat{s}_i = \frac{1}{2} \hat{\sigma}_i, \]

where \( \hat{\sigma}_i \) are the Pauli matrices.

Standard representation of the Pauli matrices:

\[
\begin{align*}
\hat{\sigma}_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\hat{\sigma}_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\hat{\sigma}_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{align*}
\]

with eigenkets (1 \( \equiv x \), 2 \( \equiv y \), 3 \( \equiv z \) axis)

\[
\begin{align*}
| >_1 &= \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right),
| >_2 &= \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 \\ i \end{pmatrix} \right),
| >_3 &= \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),
\end{align*}
\]

\[
\begin{align*}
| >_1 &= \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right),
| >_2 &= \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 \\ -i \end{pmatrix} \right),
| >_3 &= \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).
\end{align*}
\]

Form of eigenvalues of \( j^2 \) and \( j_3 \):

\[ j^2 | j, m \rangle = j(j + 1) | j, m \rangle; \ j_3 | j, m \rangle = m | j, m \rangle. \]

Here, \( j = 0, 1/2, 1, 3/2, ..., -j \leq m \leq j \), in integer steps.

Total angular momentum, if composed of two angular momenta, \( \hat{j}_1 \) and \( \hat{j}_2 \), \( \hat{j} = \hat{j}_1 + \hat{j}_2 \).

(Vectorial sum)

\[ \hat{J} \cdot \hat{J} = J^2 = (\hat{j}_1 + \hat{j}_2) \cdot (\hat{j}_1 + \hat{j}_2); [\hat{j}_1, \hat{j}_2] = 0. \]

Eigenvalues and eigenvectors of the total angular momentum:

\[ J^2 | J, M(j_1, j_2) \rangle = J(J + 1) | J, M(j_1, j_2) \rangle, \]

\[ J_3 | J, M(j_1, j_2) \rangle = M | J, M(j_1, j_2) \rangle, \]

\[ | j_1 - j_2 | \leq J \leq j_1 + j_2 \] (integer steps); \( M = m_1 + m_2 \).

\[ < j', m' | j'_+ | j, m \rangle = \delta_{j', j} \delta_{m', m + 1} [j(j + 1) - m(m + 1)]^{1/2} = \]

\[ = < j, m | j_- | j', m' > \]
Stationary perturbation theory.

\[ \mathcal{H} = \mathcal{H}_0 + \nu, \]

\[ \mathcal{H}_0 |n0> = E_{n0} |n0>, \]

\[ \mathcal{H} |n> = E_n |n>, \]

\[ E_n = E_{n0} + <n0|\nu|n0> + \Sigma'_k \frac{<n0|\nu|k0><k0|\nu|n0>}{E_{n0} - E_{k0}} + ..., \]

\[ |n> = |n0> + \Sigma'_k \frac{|k0><k0|\nu|n0>}{E_{n0} - E_{k0}} \]

(Here, the prime over the sum means that the term with \( k = n \) is to be omitted.)

Hydrogen atom. (Non-relativistic approximation.)

The bound state energy levels are given by the formula:

\[ E_n = -\frac{\mu e^4}{2\hbar^2}\frac{1}{n^2}. \]

Here \( \mu \) is the reduced mass of the electron and proton, \( n \) is the principal quantum number, \( n = 1, 2, 3, ... \)

Bohr radius:

\[ a = \frac{\hbar^2}{\mu e^2}. \]

Wave functions in the bound states:

\[ <r, \theta, \phi|n, l, m> = R_{n,l}(r)Y_l^m(\theta, \phi); \rho = \frac{r}{a} \]

The wave functions for \( l = 0 \) (S-states) are of the form:

\[ <r, \theta, \phi|n, 0, 0> = \frac{1}{\sqrt{\pi (na)^3/2}} \frac{1}{n! n^2} F_{n,0}(\frac{2\rho}{n}) \]

\[ F_{n,0}(x) = \exp \left( -\frac{x}{2} \right) \sum_{s=0}^{n-1} (-1)^s x^s \frac{(nl)^2}{s! (s+1)! (n-1-s)!} \]

Hydrogen wave functions for \( n = 1, 2 \)

\[ \psi_{n,l,m}(r, \theta, \phi) \]
\[ \psi_{1,0,0}(r, \theta, \varphi) = \frac{1}{a^{3/2}} \frac{e^{-\rho}}{\sqrt{\pi}} \]
\[ \psi_{2,0,0}(r, \theta, \varphi) = \frac{1}{a^{3/2}} \frac{1}{\sqrt{32\pi}} (2 - \rho) e^{-\frac{\rho}{2}} \]
\[ \psi_{2,1,0}(r, \theta, \varphi) = -\frac{1}{a^{3/2}} \frac{1}{\sqrt{64\pi}} \rho \sin \theta e^{i\varphi} e^{-\frac{\rho}{2}} \]
\[ \psi_{2,1,1} = -\psi_{2,1,0}^* \]
\[ \psi_{2,1,0}(r, \theta, \varphi) = \frac{1}{a^{3/2}} \frac{1}{\sqrt{32\pi}} \rho \cos \theta e^{-\frac{\rho}{2}} \]

**Scattering theory.**

Assume fixed target, elastic scattering and spin zero. The quantity \( k \) stands for the absolute value of the wave number of the incident wave, \( \theta \) is the scattering angle. If both the target and projectile are spinless and the scattering potential is spherically symmetric, the scattering is cylindrically symmetric.

Scattering amplitude, partial wave decomposition is:

\[ f(\theta) = \sum_{l=0}^{\infty} (2l + 1) \frac{\exp(i\delta_l) \sin \delta_l}{k} P_l(\cos \theta) \]

Here the \( P_l(x) \) are Legendre polynomials and \( \delta_l \) are real.

Differential cross section:

\[ \frac{d\sigma}{d\Omega} = |f(\theta)|^2, \]

where \( d\Omega = \sin \theta d\theta d\phi \), the element of solid angle.

Total cross section

\[ \sigma = \int d\Omega \frac{d\sigma}{d\Omega}. \]

The quantities \( f, \delta_l, \frac{d\sigma}{d\Omega} \) and \( \sigma \) are in general all functions of \( k \) (the wave vector) and therefore of energy.

The optical theorem is

\[ \sigma = \frac{4\pi}{k} \text{Im}f(0). \]

First Born approximation to the elastic scattering amplitude.

\[ f(\theta) \approx -\frac{m}{2\pi\hbar^2} \int d^3x \exp\{i(k - k') \cdot x\} V(x), \]

where \( k \) and \( k' \) stand for the wave vectors of the projectile in the initial and final states, respectively. In the case of elastic scattering, \( |k| = |k'| \).
Miscellaneous formulae.

Legendre polynomials.

\[ P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2} (3x^2 - 1) \quad (-1 \leq x \leq 1) \]

\[ (l + 1)P_{l+1}(x) = (2l + 1)xP_l(x) - lP_{l-1}(x). \]

Orthogonality:

\[ \int_{-1}^{1} dx P_l(x)P_l(x) = \frac{2\delta_{l,0}}{2l + 1}, \]

where \( \delta_{a,b} = 1 \) if and only if \( a = b \); it is zero otherwise. ("Kronecker's δ symbol".)

\[ \int_{0}^{\infty} x^n e^{-x} \, dx = n!. \]

A book of integrals is available if you need it.
Formulae

Laplace operator in spherical coordinates:
\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \]

A commutation relation
\[ [x, p_x] = i\hbar \]

Spherical Harmonics
\[ Y_{0,0}(\theta, \phi) = \sqrt{\frac{1}{4\pi}}, \quad Y_{1,1}(\theta, \phi) = \frac{1}{2\sqrt{2\pi}} \sin \theta, \quad Y_{1,0}(\theta, \phi) = \frac{1}{2\sqrt{2\pi}} \cos \theta \]
\[ [Y_{l,m}(\theta, \phi)]^* = (-1)^m Y_{l,-m}(\theta, \phi) \]
\[ \int d\Omega Y_{lm}(\Omega) Y_{lm}^*(\Omega) = (2l+1) \delta_{ll} \delta_{mm} \]
\[ Y_{l_1,m_1}(\Omega) Y_{l_2,m_2}(\Omega) = \sum_{l=|l_1-l_2|}^{l_1+l_2} \sum_{m=-l}^{l} \sum_{m_1}^{m_2} \chi_{l_1l_2,m_1m_2}^{l_1l_2}(\Omega) Y_{l,m} Y_{l,m_1}^*(\Omega), \]
where \[ \chi_{l_1l_2,m_1m_2}^{l_1l_2} = \frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)} \langle l_1, l_2; 0,0 | l, 0 \rangle \langle l_1, l_2; m_2, m_2 | l, m_1 + m_2 \rangle \]
and the Clebsch-Gordan coefficient, \( \langle l_1, l_2; 0,0 | l, 0 \rangle \), vanishes unless \( l_1 + l_2 - l \) is even.

Harmonic Oscillator
\[ \dot{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^*) \quad \dot{p} = i\sqrt{\frac{\hbar}{2m\omega}} (-\hat{a} + \hat{a}^*) \]
\[ [\hat{a}, \hat{a}^*] = 1 \]
\[ \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \hat{a}^*|n\rangle = \sqrt{n+1}|n+1\rangle \]

Stationary non-degenerate perturbation theory
\[ \hat{H} = \hat{H}_0 + \hat{H}_1 \]
\[ E_n = E_n^{(0)} + \langle \phi_n^{(0)} | \hat{H}_1 | \phi_n^{(0)} \rangle + \sum_{m \neq n} \frac{|\langle \phi_m^{(0)} | \hat{H}_1 | \phi_n^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}} + \ldots \]
\[ |\psi_n\rangle = |\phi_n^{(0)}\rangle + \sum_{m \neq n} |\phi_m^{(0)}\rangle \frac{\langle \phi_m^{(0)} | \hat{H}_1 | \phi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} + \ldots \]

Time-dependent perturbation theory.
\[ \hat{H} = \hat{H}_0 + \hat{H}_1(t) \]
Initial state $|\psi(0)\rangle = |E^{(0)}_k\rangle$, $|\psi(t)\rangle = \sum_n c_n(t) \exp\left(-\frac{iE^{(0)}_n t}{\hbar}\right)|E^{(0)}_n\rangle$

$c_n(t) = \delta_{nk} - \frac{i}{\hbar} \int_0^t dt' \exp\left(i(E^{(0)}_n - E^{(0)}_k)t'/\hbar\right) \langle E^{(0)}_n|\hat{H}_1(t')|E^{(0)}_k\rangle + ...$

Partial wave expansion and phase shifts

$\sigma_i(k) = \frac{4\pi}{k^2} (2l+1) \sin^2 \delta_i(k)$

$\psi(r \theta \phi) \to \sum_{i=0}^\infty C_i \frac{\sin(kr - l \frac{\pi}{2} + \delta_i(k))}{kr} P_l(\cos \theta) \text{ for } r \to \infty$

Born Approximation

$f(q) = -\frac{m}{2\pi \hbar^2} \int d^3r V(r) \exp(-iq\cdot r)$

Potentially useful mathematical results

$\int_{-\infty}^{\infty} e^{i\omega t} e^{-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2} dt = \sqrt{2\pi \sigma} e^{-i(\omega \sigma)^2}$

$\int x \sin x \, dx = \sin x - x \cos x$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$

$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ... \text{ for } |x| < \frac{\pi}{2}$
Useful Equations

- Laplacian Operator in Cylindrical Coordinates:

\[ \nabla^2 = \frac{1}{r \frac{\partial}{\partial r}} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \]

- Harmonic Oscillator

\[ \hat{a} = \sqrt{\frac{m \omega}{2 \hbar}} (\hat{x} + \frac{i}{m \omega \hbar} \hat{p}) \quad [\hat{a}, \hat{a}^\dagger] = 1 \]
\[ \hat{x} = \sqrt{\frac{\hbar}{2 m \omega}} (\hat{a} + \hat{a}^\dagger) \quad \hat{p} = -i \sqrt{\frac{m \omega \hbar}{2}} (\hat{a} - \hat{a}^\dagger) \]
\[ \hat{a} |n\rangle = \sqrt{n} |n - 1\rangle \quad \hat{a}^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle \]
\[ |n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle \]

- Stationary Non-degenerate Perturbation Theory

\[ \hat{H} = \hat{H}_0 + \hat{H}_1 \]
\[ E_n = E_n^{(0)} + \langle \phi_n^{(0)} | \hat{H}_1 | \phi_n^{(0)} \rangle + \sum_{m \neq n} \frac{\left| \langle \phi_m^{(0)} | \hat{H}_1 | \phi_n^{(0)} \rangle \right|^2}{E_n^{(0)} - E_m^{(0)}} + \ldots \]
\[ |\psi_n\rangle = |\phi_n^{(0)}\rangle + \sum_{m \neq n} |\phi_m^{(0)}\rangle \frac{\langle \phi_m^{(0)} | \hat{H}_1 | \phi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} + \ldots \]
35. Clebsch-Gordan coefficients, Spherical Harmonics, and d Functions

Note: A square-root sign is to be understood over every coefficient, e.g., for $-8/15$ read $\sqrt{-8/15}$.

\[ Y^0 = \sqrt{3/4} \cos \theta \]
\[ Y^1 = -\sqrt{3/8} \sin \theta e^{i\phi} \]
\[ Y^2 = \sqrt{5/16} \left( \cos^2 \theta - 1/2 \right) \]
\[ Y_1 = -\sqrt{5/8} \sin \theta \cos \phi e^{i\phi} \]
\[ Y_2 = \frac{1}{4} \sqrt{5/2} \sin^2 \theta e^{i\phi} \]

\[ Y_{l,m} = (-1)^m Y_l^m \]
\[ d_{l,m,0} = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{array} \right) \]
\[ d_{l,m,1} = \frac{1 + \cos \theta}{\sqrt{2}} \]
\[ d_{l,m,-1} = \frac{\sin \theta}{\sqrt{2}} \]

\[ d_{4/2,3/2} = \frac{1 + \cos \theta}{2} \]
\[ d_{3/2,1/2} = -\sqrt{3} \frac{1 - \cos \theta}{\sin \theta} \]
\[ d_{3/2,-1/2} = \frac{\sqrt{5}}{2} \sin^2 \theta \]
\[ d_{3/2,-3/2} = -\frac{3 \cos \theta - 1}{2} \sin \theta \]

Figure 35.1: The sign convention is that of Wigner (Group Theory, Academic Press, New York, 1959), also used by Condon and Shortley (The Theory of Atomic Spectra, Cambridge Univ. Press, New York, 1935), Rose (Elementary Theory of Angular Momentum, Wiley, New York, 1957), and Cohen (Tables of the Clebsch-Gordan Coefficients, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.
Problem 1. (20 points)
n_1 \text{ moles of a monatomic gas and } n_2 \text{ moles of a diatomic gas are mixed together in a container.}

(a) Derive an expression for the molar specific heat at constant volume of the mixture (10 points)
(b) Show that your expression has the expected behavior if \( n_1 \rightarrow 0 \) or \( n_2 \rightarrow 0 \) (5 points)
(c) What fraction of the gas needs to be monatomic so that \( \gamma = \frac{c_p}{c_v} = 1.5 \) ? (5 points)

Problem 3. (25 points)
An elastic rubber string can be stretched to length \( L \) with a force \( P \), described by the relation \( P = A(L) T \). The function \( A(L) \) is a monotonically increasing function of \( L \), determined experimentally.

(a) Show that the internal energy only depends on the temperature \( T \), and that the entropy is a function of \( L \) only (10 points)
(b) Show that the temperature of the string increases if the string is stretched adiabatically (15 points)

Problem 4. (20 points)
Consider an infinite vertical cylinder with cross section \( A \), containing air with a temperature \( T \) (treat the air as a monatomic ideal gas). Assume that the cylinder is in a constant gravitational field. Calculate the specific heat of the system (\( C_v \)). Hint: Use the barometric formula:
\[
\rho(z) = \rho(0) \exp\left(-\frac{gz}{kT}\right)
\]

Problem 5. (15 points)
A small, ideal black body has a surface area of 1 cm\(^2\). It is located in the center of a larger box, also a black body, containing vacuum. The large box has a surface temperature of 300 °K. How much power is needed (in W) to keep the small body at a fixed temperature of 1000 °K?

2. A reversible heat engine operates between two reservoirs: a very large one at a temperature \( T_1 \) and a smaller, warmer one at a temperature \( T_2 \). After the engine has operated for some long period of time, the temperature of the warmer reservoir has been lowered to \( T_1 \), while the temperature of the larger reservoir remained the same. Assume that the warmer reservoir consists of \( n \) moles of ideal gas at a constant volume and is characterized by a specific heat capacity \( c_v \).

a) Find the heat extracted from the warmer reservoir.
b) Find the change in entropy for the warmer reservoir.
c) Find the total work done by the engine in this time period.
3. The three lowest energy levels of a certain molecule are \( E_1=0 \), \( E_2=e \) and \( E_3=10e \). Assume that the system consists of \( N \) particles and obeys Maxwell-Boltzmann statistics.
   a) Write an expression for the temperature-dependent occupation of level \( E_3 \), \( N_3(T) \), and find the temperature \( T_C \) below which \( N_3(T)<1 \). Assume that \( N>>1 \).
   b) Find the average energy \( E \) of the molecule at temperature \( T \).
   c) Find the contributions of these levels to the specific heat per mole \( c_V \).
   d) Sketch \( c_V \) as a function of \( T \).

4. Our universe is pervaded with black body photon radiation at \( T=3K \), which is believed to have come from the adiabatic expansion of a much hotter photon cloud produced during the Big Bang.
   a) Why is the expansion adiabatic, rather than, for example, isothermal?
   b) If in the next \( 10^{10} \) years the volume of the universe increases by a factor of two, what will then be the temperature of the black body radiation?
   c) Write down an integral which determines the density of the energy (energy per cubic meter) contained in this cloud of radiation. Estimate the result within an order of magnitude in joules/(meter)\(^3\).

1. You are given \( N \) true dice, each with six numbered faces \( \{1,2,3,4,5,6\} \)
   a) What is the probability for obtaining a number greater than four when rolling a single die? (2 points)
   b) I now roll all \( N \) dice. Obtain an algebraic expression for the probability of obtaining \( M \) dice each with a number greater than four. (6 points)
   c) I again roll all \( N \) dice. Obtain an algebraic expression for the probability that the first \( M \) dice rolled will each be a number greater than four, and the last \( (N-M) \) dice will each be a number less than or equal to four. (6 points)
   d) I again roll all \( N \) dice. Obtain an expression for the expected mean total of the values on all dice. (6 points)

**Problem 1 (15 points)**

A sealed volume is separated into two chambers \( A \) and \( B \) with volumes \( V_A \) and \( V_B \) respectively. There is a thermally insulating barrier surrounding the entire volume and the separating wall between volumes \( A \) and \( B \) is thermally insulating. Initially, there is one mole of \( N_2 \) in \( A \) at temperature \( T_A \) and one mole of \( Ar \) in \( B \) at temperature \( T_B \). Both gases may be treated as ideal. The barrier is now removed allowing the two gasses to develop a common thermodynamic equilibrium.

(a) Determine the final temperature of the mixed gas. (5 points)

(b) Determine the change in entropy that resulted from removing the barrier. (5 points)

(c) Repeat (a) and (b) starting with 1 mole of \( Ar \) in both chambers at start temperatures \( T_A \) and \( T_B \). (5 points)
Problem 4

The DNA molecule forms the much heralded "double helix" structure where two long sugar phosphate strands are inter-bonded at regular intervals by amino acid base pairs (See Fig. 1, left). In order to share its genetic information, the DNA molecule must "unzip" in a process where the bonds between strands are broken in a sequential fashion. The statistics of thermally active unbinding can be roughly captured by the so-called single-ended zipper model (See Fig. 1, right). Consider a model of DNA as a quantum zipper with $N$ parallel links that can be opened only from one end. Within this model if links $1, 2, \ldots, p$ are open then it takes energy $\varepsilon$ to open link $p+1$. If all the preceding links are not open, the energy required to open the $p+1$ link is infinite. In the single-ended model, the energy to open the final link $p=N$ is also infinite. After a link is open there are $G$ different configurations the link can then take. This is related to the rotational freedom of a link.

(a) How many different configurations exist with energy $p\varepsilon$?

(b) What is the partition function of an $N$-link strand of DNA within the single ended zipper model?

(c) What is the thermodynamic average of open links? (Hint: In order to get the relevant series sum consider the derivative of $\ln Z$.) Derive an approximate expression at low temperature. This result is independent of DNA strand length. Why?

![Figure 1](image)

Figure 1. (left) Schematic of the DNA double helix showing base pair bonding between the sugar phosphate backbone. (right) The thermodynamics of DNA can be captured by the single ended zipper model, where base pair bonding is represented as links on the quantum zipper.

Problem 2 (14 points)

$N$ distinguishable atoms in a solid can be in one of two states with energy $\theta$ and $\varepsilon$.

(a) How many ways are there for the system to have a specific energy $E=m\varepsilon$. (5 points)

(b) Use Stirling's formula to write an expression for the $E$-dependence of the entropy. (5 points)

(c) Use $\frac{1}{T} = \frac{\partial S}{\partial E}$ to calculate the temperature of the system as a function of $E$. (4 points)
5. Slides and springs

A massless spring (spring constant $\kappa$) connects two small sliding masses $m$. In the initial configuration, the masses are at $x_1 = a$, $x_2 = 2a$, and the spring is relaxed (Figure 1). The walls at 0 and 3a are fixed. The masses are set on a rod, so that they can move only along the $x$ axis. The system is placed into a thermal reservoir maintained at a temperature $T$.

(a) Write down the total energy (Hamiltonian) of this system in the coordinates $x_1$, $x_2$ (positions of the masses), $p_1$, $p_2$ (momenta of the masses).

(b) The partition function is

$$Z = \int e^{-\beta H(x_1,p_1)} dx_1 dx_2 dp_1 dp_2 / h_0^N,$$

where $\beta = 1/kT$, $N = 2$ and $h_0$ is the grid size of the phase space. What are the appropriate limits of integration for our system? Evaluate the partition function assuming that the spring is very weak, so if you use the Taylor expansion, you only need to take into account the lowest non-zero order in $\kappa$.

(c) Give the general formula for the relationship between the mean energy and the partition function for a system in contact with a thermal reservoir. Use this formula to calculate the mean energy of the masses-and-spring system, including the leading term in $\kappa$. What is the mean energy per degree of freedom in the limit $\kappa = 0$ (no spring)?

Figure 1
Problem 3

In a ferromagnet, the excitations are normal modes called magnons. Magnons obey Bose-Einstein statistics, and we can approximate their density of states as:

\[
g(\omega) = \begin{cases} 
C\omega^{D/2-1} & \omega \leq \omega_{\text{max}} \\
0 & \omega > \omega_{\text{max}}
\end{cases}
\]

where \( D \) is the spatial dimension of the ferromagnet.

(a) For \( D = 3 \) find the temperature dependence of the magnon specific heat at low temperature (\( k_B T \ll \hbar \omega_{\text{max}} \)). Do not worry about numerical prefactors.

(b) The magnetization of a ferromagnet is suppressed from its \( T = 0 \) limit by an amount proportional to the total number of magnons of all energies. Show that for \( D = 3 \) the magnetization at low temperatures has a form:

\[
M(T) = M_0 - AT^{3/2}
\]

where \( A \) is a constant.

(c) Show that the number diverges above \( T = 0 \) for \( D \leq 2 \), and thus that the magnetization is zero at any temperature above \( T = 0 \).

4. In the 18 century the existence of binary stars (two stars in orbit around each other) was deduced by John Michell using statistics before the phenomenon was directly observed. Here was his argument: there were \( \sim 1000 \) known stars at the time yet some pairs, like the two bright stars in the constellation Capricornus \( \beta_1 \) and \( \beta_2 \), were only 0.05 degrees apart.

   a) Given that the sky has \( \sim 40,000 \) square degrees, if \( \beta_1 \) and \( \beta_2 \) are randomly placed on the sky, what are the odds they would be more than 0.05 degrees apart? (odds are a fraction or decimal > 1) (5 points)

   b) Now, if all known 1000 stars (at the time) were randomly sprinkled on the sky, what are the odds that the closest pair would be more than 0.05 degrees apart? Use \( (1-x)^n = 1-nx+... \) for \( x<<1 \). (10 points)

   c) Therefore what could John Michell conclude? (5 points)

5. Electromagnetic radiation at temperature \( T \), fills a cavity of volume \( V \). If the volume of the thermally insulated cavity is expanded quasi-statically to a volume \( 8V \), what is the final temperature \( T_f \)? (Neglect the heat capacity of the cavity walls) (20 points)
Problem 4 (12 points)

One mole of an ideal monatomic gas is the working substance in a heat engine. Starting at pressure $p_0$ and volume $V_0$, the gas goes through an isobaric expansion by a factor 4, isochoric cooling to the initial temperature, and then an isothermal compression back to the initial state. This process is shown in Fig. 1.

(a) For each leg of the process, calculate the change in internal energy $\Delta E$, the change in entropy $\Delta S$, the heat flow to the gas $\Delta Q$, and the work done by the gas, $\Delta W$. (6 points)

(b) Determine the efficiency of the engine for converting heat into work and compare the result to the efficiency of a Carnot cycle operating between the maximum and minimum temperatures of the process. (6 points)
Problem 5

Consider a substance that displays both liquid and crystal phases to arbitrarily low temperature (He$^4$ is an example of such a material). The boundary between the liquid and crystal phases, shown in the diagram below, approaches $T = 0$ at a pressure $P_0$ with essentially zero slope (i.e., $dP/dT = 0$). The crystal has higher density than the liquid. Take the volume of one mole of the crystal to be $V_C$ and one mole of the liquid to be $V_L$ with $V_L > V_C$.

(a) What is the ratio of the chemical potentials of the two phases at $P_0$ in the limit of low temperature?

(b) What is the difference in the entropies of one mole of crystal and one mole of liquid at $P_0$ in the limit of low temperature?

(c) What is the difference in the internal energies of one mole of crystal and one mole of liquid at $P_0$ in the limit of low temperature?

![Diagram showing the phase boundary between liquid and crystal phases with pressure on the y-axis and temperature on the x-axis. The boundary approaches $T = 0$ at $P_0$.]
Information Sheet

\[ \text{Helmholtz free energy} \quad F = E - TS \]
\[ \text{Enthalpy} \quad \mathcal{H} = E + PV \quad \text{(for hydrostatic case)} \]
\[ \text{Gibbs free energy} \quad G = E - TS + PV \quad \text{(for hydrostatic case)} \]

\[ F = -k_B T \ln Z \]

\[ \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} \, dx = 1 \]

\[ \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-(x-x)^2/2\sigma^2} \, dx = x^2 + \sigma^2 \]
\[ \ln N! \equiv N \ln N - N \]

\[ \lim_{N \to \infty} \left(1 + \frac{x}{N}\right)^{N} = e^{x} \]

\[ e^{x} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n} \]

\[ p(n_{R}) = \frac{N!}{(N-n_{R})! n_{R}!} p_{n_{R}}^{N-n_{R}} \]

\[ p(n) = \frac{1}{\sqrt{2\pi \sigma^{2}}} \exp \left(- \frac{(x-x^{2})}{2\sigma^{2}} \right) \]

\[ \int_{-\infty}^{\infty} e^{-a^2 x^2} \, dx = \frac{\sqrt{\pi}}{a} \]

\[ \text{volume of an } M\text{-dimensional sphere of radius } R = \frac{\pi^{M/2}}{(M/2)!} R^{M} \]

\[ (1 + x)^{n} = 1 + nx + \frac{n(n-1)}{2!} x^{2} + \frac{n(n-1)(n-2)}{3!} x^{3} + \cdots + \frac{n!}{(n-r)! r!} x^{r} + \cdots. \]

Note that here and elsewhere we take 0! = 1. If \( n \) is a positive integer, the expression consists of a finite number of terms. If \( n \) is not a positive integer, the series is convergent for \( x^{2} < 1 \); and if \( n > 0 \), the series is convergent also for \( x^{2} = 1 \). [Ref. 21, p. 88.]

\[ (1 + x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^{2} - \frac{n(n+1)(n+2)}{3!} x^{3} \]

\[ + \cdots + (-1)^{r} \frac{(n+r-1)!}{(n-1)! r!} x^{r} + \cdots, \quad [x^{2} < 1]. \]

\[ e^{x} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots + \frac{x^{n}}{n!} + \cdots, \quad [x^{2} < \infty]. \]

**Information Sheet**

\[ PV = Nk_{B}T \quad \text{'ideal gas law'} \]

\[ \int_{0}^{\infty} x e^{-x^{2}} \, dx = \frac{1}{2} \]

\[ \int_{0}^{\infty} x^{2} e^{-x^{2}} \, dx = \frac{\sqrt{\pi}}{4} \]

\[ p(n; L) = \frac{1}{n!} (\pi r L)^{n} e^{-\pi r L} \]

\[ (x + y)^{N} = \sum_{n=0}^{N} \frac{N!}{n! (N-n)!} x^{n} y^{N-n} \]

\[ \cosh x = \frac{1}{2} (e^{x} + e^{-x}) \]

\[ \sinh x = \frac{1}{2} (e^{x} - e^{-x}) \]

\[ \tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \]

\[ \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}} \]
Formulae

\[ \ln N! \approx N \ln N - N \]

\[ \binom{N}{m} = \frac{N!}{m!(N-m)!} \]

\[ pV = Nk_B T \]

\[ C_v = N \frac{f}{2} k_B \]

\[ S = Nk_B \left[ \ln \frac{V}{N} + \frac{C_v}{Nk_B} \ln T + \sigma_0 \right] \]

\[ \left( \frac{\partial x}{\partial y} \right)_z = \frac{\frac{\partial (\partial z)}{\partial y}}{\left( \frac{\partial z}{\partial x} \right)_y} \]

\[ \left( \frac{\partial x}{\partial y} \right)_w = \frac{\left( \frac{\partial x}{\partial w} \right)_z}{\left( \frac{\partial x}{\partial y} \right)_w} \]

\[ \left( \frac{\partial x}{\partial y} \right)_z = \frac{1}{\left( \frac{\partial y}{\partial x} \right)_z} \]

\[ F_G = G \frac{m_1 m_2}{R^2} \]

Geometric series sums:

\[ \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r} \]

\[ \sum_{s=0}^{\infty} se^{-sv} = -\frac{d}{dy} \sum_{s=0}^{\infty} e^{-sv} = -\frac{d}{dy} \left[ \frac{1}{1 - e^{-y}} \right] = \frac{e^{-y}}{(1 - e^{-y})^2} \]
Avogadro number $N_A = 6.02 \times 10^{23}$ mole$^{-1}$.

Boltzmann constant $k_B = 1.38 \times 10^{-16}$ erg K$^{-1}$.

Electron mass $m_e = 9.11 \times 10^{-28}$ g.

Plank's constant $\hbar = 1.05 \times 10^{-27}$ erg s.

Thermal occupation numbers for ideal bosons and fermions:

$$N_B(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_BT} - 1}, \quad N_F(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_BT} + 1},$$

where $\epsilon$ is the energy and $\mu$ is the chemical potential.

Bose integrals:

$$\int_0^\infty \frac{x^{s-1} \, dx}{e^x - 1} = \int_0^\infty dx \, x^{s-1} \sum_{n=1}^\infty e^{-nx} = (s-1)! \zeta(s), \quad \text{where} \quad \zeta(s) = \sum_{n=1}^\infty n^{-s}.$$

Numerical values of the Riemann zeta function:

$$\zeta(2) = \frac{\pi^2}{6} \approx 1.645, \quad \zeta(3) \approx 1.202, \quad \zeta(4) = \frac{\pi^4}{90} \approx 1.082, \quad \zeta(5) \approx 1.037.$$

First Law of Thermodynamics:

$$dQ = dU + dW \quad (1)$$

$$F = -k_BT \ln Z \quad \mu = \left[ \frac{\partial F}{\partial N} \right]_{T,V}$$

where $F$ is the free energy and $Z$ is the partition function

$$\frac{1}{k_B} \left[ \frac{\partial F}{\partial V} \right]_T = -\sigma \quad \left[ \frac{\partial F}{\partial V} \right]_T = -p$$

where $\sigma$ is the entropy and $p$ is the pressure.

$$\Delta S = \int_1^T \frac{dQ}{T}$$

where $\Delta S$ is the entropy change, $Q$ is the heat, and $T$ is temperature.

$$W_{1\rightarrow 2} = \int_1^2 pdV \quad pV = nRT$$

where $W_{1\rightarrow 2}$ is the work done on a path from $1 \rightarrow 2$, $p$ is pressure, $V$ is volume, $n$ is the number of moles, $R$ is the ideal gas constant and $T$ is temperature.

$$\frac{\pi^2}{15} = \int_0^\infty \frac{x^3 \, dx}{e^x - 1} \quad \frac{\sqrt{\pi}}{2} = \int_0^\infty xe^{-x^2}$$
Every problem is worth 20 points.
Attempt to solve all problems.
You may use a calculator.
Books and notes are not allowed.

Potentially useful formulae:

Avogadro number $N_A = 6.02 \times 10^{23}$ mole$^{-1}$.
Boltzmann constant $k_B = 1.38 \times 10^{-16}$ erg K$^{-1}$.
Electron mass $m_e = 9.11 \times 10^{-28}$ g.
Plank's constant $\hbar = 1.06 \times 10^{-27}$ erg s.

Thermal occupation numbers for ideal bosons and fermions:

$$N_b(\epsilon) = \frac{1}{e^{(\epsilon - \mu)/k_B T} - 1}, \quad N_f(\epsilon) = \frac{1}{e^{(\epsilon - \mu)/k_B T} + 1},$$

where $\epsilon$ is the energy and $\mu$ is the chemical potential.

Bose integrals:

$$\int_0^\infty \frac{x^{s-1} dx}{e^x - 1} = \int_0^\infty dx \frac{x^{s-1} e^{-nx}}{n^s} = (s-1)! \zeta(s), \quad \text{where } \zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

Numerical values of the Riemann zeta function:

$$\zeta(2) = \frac{\pi^2}{6} \approx 1.645, \quad \zeta(3) \approx 1.202, \quad \zeta(4) = \frac{\pi^4}{90} \approx 1.082, \quad \zeta(5) \approx 1.037.$$
Formulae

\[ \ln N! \approx N \ln N - N \]

Stirling’s formula for large \( N \)

\[ \binom{N}{m} = \frac{N!}{m!(N-m)!} \]

Binomial coefficient

\[ pV = Nk_B T \]

Equation of state for ideal gas

\[ C_v = N \frac{f}{2} k_B \]

Ideal gas of \( N \) molecules each with \( f \) degrees of freedom

\[ S = Nk_B \left[ \ln \frac{V}{N} + \frac{C_v}{Nk_B} \ln T + \sigma_0 \right] \]

Entropy of ideal gas

\[ \left( \frac{\partial x}{\partial y} \right)_z = -\frac{\left( \frac{\partial z}{\partial y} \right)_x \left( \frac{\partial z}{\partial x} \right)_y}{\left( \frac{\partial z}{\partial x} \right)_x \frac{\partial z}{\partial y} \left( \frac{\partial z}{\partial y} \right)_x} \]

Partial derivatives

\[ \left( \frac{\partial x}{\partial y} \right)_z = \frac{\left( \frac{\partial x}{\partial w} \right)_z \left( \frac{\partial y}{\partial w} \right)_z}{\left( \frac{\partial y}{\partial w} \right)_z} \]

\[ \left( \frac{\partial x}{\partial y} \right)_z = \frac{1}{\left( \frac{\partial y}{\partial x} \right)_z} \]

\[ F_0 = G \frac{m_1 m_2}{R^2} \]
FIG 2. (left) Schematic of the DNA double helix showing base pair bonding between the sugar phosphate backbone. (right) The thermodynamics of DNA can be captured by the single ended ‘zipper’ model, where base pair bonding is represented as links on the quantum zipper.

Potentially useful constants and expressions:

Avogadro’s number: \( N_A = 6.02 \times 10^{23} \text{ mole}^{-1} \)

Ideal gas constant: \( R = 8.3 \text{ J K}^{-1} \text{ mol}^{-1} \)

Boltzmann constant: \( k_B = 1.38 \times 10^{-23} \text{ J K}^{-1} \)

\[ R = N_A k_B \]

One mole of an ideal gas occupies 22.4 liters at standard temperature and pressure.

Electron mass: \( m_e = 9.11 \times 10^{-28} \text{ g} \)

Bose-Einstein and Fermi-Dirac distribution functions:

\[ N_b = \frac{1}{e^{\frac{\epsilon - \mu}{k_B T}} - 1} \quad N_f = \frac{1}{e^{\frac{\epsilon - \mu}{k_B T}} + 1} \]

where \( \epsilon \) is the energy and \( \mu \) is the chemical potential.

First Law of Thermodynamics:

\[ dQ = dU + dW \]

Thermodynamic identity

\[ \tau d\sigma = dU + p dV \]

\( \tau \) is \( k_B T \).

Helmholtz Free Energy

\[ F = U - \tau \sigma \]

\[ F = -k_B T \ln Z \quad \mu = \left[ \frac{\partial F}{\partial N} \right]_{T,V} \]

where \( F \) is the free energy, \( Z \) is the partition function, \( \mu \) is the chemical potential, and \( N \) is the number of particles.

\[ U = \tau^2 \frac{\partial \ln Z}{\partial \tau} \]
where $U$ is the average system energy

$$\frac{1}{k_B} \left[ \frac{\partial F}{\partial T} \right]_V = -\sigma \quad \left[ \frac{\partial F}{\partial V} \right]_T = -p$$

where $\sigma$ is the entropy and $p$ is the pressure.

$$\Delta S = \int_1^2 \frac{dQ}{T}$$

where $\Delta S$ is the entropy change from state $1 \rightarrow 2$, $Q$ is the heat, and $T$ is temperature.

$$W_{1 \rightarrow 2} = \int_1^2 pdV \quad pV = nRT$$

where $W_{1 \rightarrow 2}$ is the work done on a path from $1 \rightarrow 2$, $p$ is pressure, $V$ is volume, $n$ is the number of moles, $R$ is the ideal gas constant and $T$ is temperature.

$$\frac{\pi^2}{15} = \int_0^\infty dx \frac{x^3}{e^x - 1} \quad \frac{\sqrt{\pi}}{2} = \int_0^\infty dx e^{-x^2}$$

Geometric series sums:

$$\sum_{k=0}^n r^k = \frac{1 - r^{n+1}}{1 - r}$$

$$\sum_{s=0}^\infty se^{-sy} = -\frac{d}{dy} \sum_{s=0}^\infty e^{-sy} = -\frac{d}{dy} \left[ \frac{1}{1 - e^{-y}} \right] = \frac{e^{-y}}{[1 - e^{-y}]^2}$$

Numerical values of the Riemann zeta function:

$$\zeta(2) = \frac{\pi^2}{6} \approx 1.645 \quad \zeta(3) \approx 1.202 \quad \zeta(4) = \frac{\pi^4}{90} \approx 1.082 \quad \zeta(5) \approx 1.037$$

Numerical values log values:

$$\ln(2) = 0.693 \quad \ln(3) = 1.099 \quad \ln(4) = 1.38 \quad \ln(5) = 1.61$$

Stirling approximation:

$$\ln N! = N \ln N - N$$
Theoretical Mechanics

2. [20 points] You have a cylinder of radius $R$ and mass $M$ which is not homogeneous. You do however know that the mass distribution has cylindrical symmetry. You need to measure the moment of inertia of the cylinder in a simple experiment.

The cylinder is at rest on a flat horizontal table on which it can roll without slipping. The axis of the cylinder is attached by two identical springs (with spring constant $k$) to a vertical barrier that is fixed to the table. The springs are parallel with the table at all times. You have a stop-watch and can measure the period of small oscillations, $\tau$, of the system. The oscillations need to be small enough to ensure the cylinder rolls without slipping.

(a) [2 points] What is the effective spring constant, $k_{eff}$ for two springs connected in parallel?

(b) [8 points] Write the Lagrangian for the cylinder + springs (using $k_{eff}$) and derive Euler-Lagrange equations of motion.

(c) [3 points] Express the moment of inertia $I_{CM}$ about the axis of the cylinder as a function of $M$, $R$, $k_{eff}$, and $T$.

(d) [3 points] As you have seen, you don't need to measure the amplitude of the oscillation to obtain $I_{CM}$. The amplitude only comes into play because it has to be smaller than some critical amplitude $x_{max}$ to ensure that the cylinder does not slip. If the coefficient of static friction is $\mu_s$, what is the value of $x_{max}$? [Note: use Newton's laws if that is easier.]

(e) [4 points] If the table is not completely horizontal, how does that influence the measurement? Adjust the result of part (c) assuming that the table has been tilted by a small angle $\theta$. 

$\begin{align*}
\text{(a)-(d)} & \quad \begin{array}{c}
g \\ k \\ R
\end{array} \\
\text{(e)} & \quad \begin{array}{c}
\text{table tilted by angle } \theta
\end{array}
\end{align*}$
5. [20 points] A two-atom molecule can be modeled as two point particles interacting with an attractive force \( \vec{F} = -k\vec{r} \) where \( \vec{r} \) is the position vector of one particle with respect to the other. The reduced mass of the system is \( \mu \). The angular momentum of the system, \( \vec{\mathbf{L}} \), is conserved and its magnitude is known.

(a) [5 points] Make a sketch showing the potential energy \( U(r) \), the centrifugal potential energy \( U_{cf}(r) \) and the effective potential energy \( U_{eff}(r) \).

(b) [5 points] Find the equilibrium separation \( r_0 \) (the distance at which the two atoms can circle around each other with constant distance \( r = r_0 \)).

(c) [10 points] Find the frequency of small oscillations about the circular orbit if the particles are disturbed a little from the separation \( r_0 \). [Hint: expand \( U_{eff}(r) \) in a Taylor series about the equilibrium point \( r_0 \) and neglect all terms of order \( (r - r_0)^3 \) and higher.]

2. A particle is projected vertically upward in a constant gravitational field with an initial velocity \( v_0 \). Show that if there is a retarding force proportional to the square of the instantaneous velocity, then the velocity of the particle when it returns to the initial position is

\[
\frac{v_0 v_t}{\sqrt{v_0^2 + v_t^2}}
\]

where \( v_t \) is the terminal velocity.

(Potentially useful formula: \( \int \frac{x dx}{A + Bx^2} = \frac{1}{2B} \arctan \left( \frac{Bx}{\sqrt{A}} \right) \))

4. A particle moves in a plane under the influence of a force \( \vec{F} = -Ar^{\alpha-1} \) directed toward the origin; \( A \) and \( \alpha \) (\( \neq 0 \) or 1) are constants. Choose appropriate generalized coordinates and let the potential be zero at the origin.

a) Find the Lagrangian equations of motion.

b) Is the angular momentum about the origin conserved?

c) Is the total energy conserved?
4. Consider the motion of a particle of mass \( m \) in one dimension. The particle is subject to the potential

\[ V(x) = \frac{a}{x^2} + bx^2. \]

with \( a, b > 0 \), and \( x > 0 \).

(a) What is the equilibrium position of the particle?

(b) What is the restoring force on the particle as it moves a distance \( \Delta x \) away from equilibrium?

(c) What is the frequency of small oscillations about the equilibrium point?

(d) What is the first nonlinear term for small oscillations about equilibrium? For what amplitude of oscillation is the nonlinear term as important as the harmonic term?

(e) Does the nonlinear term shift the frequency of oscillation to first order in perturbation theory? Does it shift the time averaged position? If so, in which direction does it move?

5. Consider two metal spheres, each of mass \( M \) and radius \( R \). One sphere has a hollow core, of radius \( R_0 < R \).

(a) Compute the inertia tensor for each sphere.

(b) One way to tell the spheres apart is to roll them down an inclined plane. Which sphere rolls faster? What is the ratio of their velocities, as they cross any given point of the plane?
2. [20 points] Consider the Lagrangian

\[ \mathcal{L} = \frac{1}{2} m (\dot{x}^2 - \omega^2 x^2) e^{-\gamma t} \]

for the motion of a particle of mass \( m \) along one dimension (\( x \)). The constants \( m, \omega \) and \( \gamma \) are real and positive.

(a) [8 points] Find the (second order, differential) equations of motion by applying the Euler-Lagrange equations to this Lagrangian.

(b) [2 points] Interpret the equations of motion by stating the kinds of force to which the particle is subject.

(c) [5 points] Obtain the generalized momentum, and construct the Hamiltonian

\[ \mathcal{H} = px - \mathcal{L}. \]

Is the Hamiltonian a constant of the motion? Is the energy conserved? Please explain.

(d) [5 points] Obtain \( x(t) \) by trying the solution \( x(t) \sim e^{i \Omega t} \). Use \( x(0) = 0 \) and \( \dot{x}(0) = v_0 \) as initial conditions. Sketch \( x(t) \).

4. [20 points] A top with mass \( M \), moments of inertia \( \lambda_1, \lambda_2 = \lambda_1 \) and \( \lambda_3 \), is hanging from a pivot attached to the ceiling. The distance from the pivot to the center of mass is \( R_{CM} \). The top is undergoing a steady precession, and the angle \( \alpha \) between the axis of symmetry of the top and the negative \( z \) axis is constant.

(a) [15 points] If the conditions are set up so that (unknown) \( \vec{\Omega} \) is always in the horizontal plane (the plane of the ceiling), what is the precession frequency, \( \Omega_z \), of the precession about the \( z \) axis?

(b) [5 points] Can \( \vec{\Omega} \) always be in the horizontal plane if top were performing a steady precession above it, e.g., if the same top were spinning on the surface of the table?
5. [20 points] A flat uniform rectangle with sides of length \( a \) and \( b \) is at rest (i.e. without either rotational or translational motion). The origin of the coordinate system is placed at the rectangle's center, so that \( x \) is parallel to \( a \) and \( y \) is parallel to \( b \).

At \( t = 0 \), the upper left and lower right corner are simultaneously struck -- a force of magnitude \( F \) acts at both corners but in opposite directions: toward us in the upper left \((F\hat{z})\) and away from us in the lower right corner \((-F\hat{z})\), for a very brief time interval \( \Delta t \).

(a) [2 points] While the forces are acting on the corners, what is the the torque \( \vec{\tau} \) on the center-of-mass of the rectangle?

(b) [3 points] What is the angular momentum \( \Delta \vec{L} \) transferred to the rectangle after \( \Delta t \)?

(c) [10 points] What is the tensor of inertia of the rectangle in this coordinate system?

(d) [5 points] Show that the resulting initial angular velocity vector, \( \vec{\omega}(t = \Delta t) \) is oriented along the other diagonal of this rectangle.

[Hint: once you obtain \( \vec{\omega} \), you can again act on it with the tensor of inertia, \( \hat{\mathbf{I}} \), and make sure you get \( \Delta \vec{L} \).]

2) A bead can run smoothly along a rod, which swings back and forth in a vertical plane with its angle from the horizontal \( \theta = \theta_0 \cos(\Omega t) \). Write down the Hamiltonian for the bead's motion and the associated Hamiltonian equations of motion. Identify the canonical momenta for the bead; which, if any, are conserved? Is the energy conserved?

3) A freely-falling rigid body has a moment of inertia tensor relative to its center of mass \( \mathbf{I} \) and rotational velocity \( \vec{\omega} \).

(a) What is the rate of change of \( \mathbf{I} \cdot \vec{\omega} \)? That is, derive (by a simple argument) the Euler equations.

(b) If two of the principal moments of inertia are equal, find the rate of spin about the symmetry axis and the precession rate around the direction \( \hat{z} \) given the angular momentum \( \vec{J} = J\hat{z} \) and the angle \( \theta \) between the symmetry axis and \( \hat{z} \).

4) Consider a Compton scattering event in which an electron with velocity \( \vec{\beta} \) in the lab frame collides with a photon of energy \( \epsilon_i \) and direction \( \hat{n} \). Find the outgoing energy of the photon in terms of the incoming properties of the electron and photon and the outgoing direction \( \hat{n}' \) of the photon.
Useful formulas

Moments of inertia:

- homogeneous disk of radius $R$ and mass $M$: $I = \frac{1}{2}MR^2$.
- homogeneous ball of radius $R$ and mass $M$: $I = \frac{2}{5}MR^2$.
- homogeneous rectangle with lengths of sides $a$ and $b$: $I = \frac{1}{12}M(a^2 + b^2)$.

- The “parallel axis theorem”:

$$\{I\} = \{I_{CM}\} + \{I_R\}$$

where $\vec{R} \equiv (X, Y, Z)$ is the radius vector of the new origin drawn from the center of mass (CM), and $\{I_{CM}\}$ is the tensor of inertia computed in that frame. Here, the elements of $\{I_R\}$ are: $\{I_R\}_{XX} = M(Y^2 + Z^2)$, $\{I_R\}_{XY} = -MXY$, etc, doing all permutations of the coordinates $X \rightarrow Y \rightarrow Z$.

- If a vector $\vec{A}$ is constant in a frame rotating with angular velocity $\vec{\Omega}$, then it’s rate of change with respect to the lab frame is

$$\frac{d\vec{A}}{dt} = \vec{\Omega} \times \vec{A}$$

- $\int \frac{dx}{x} = \ln x + C$
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
- $\int \sin x dx = -\cos x + C$ and $\int \cos x dx = \sin x + C$
- $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
- $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
- $\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha)$
- $\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$
- Solutions of the quadratic equation $ax^2 + bx + c = 0$ are

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Scalar product: $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$.
- A second order linear differential equation $\frac{d^2 x}{dt^2} + \omega^2 x$ has the solution in the form of $x(t) = A \sin(\omega t) + B \cos(\omega t)$.