

Well-Posedness and Smoothing Effect for Nonlinear Dispersive Equations

Yoshio TSUTSUMI (Kyoto University)*

(Revised on March 21, 2018)

1. Introduction

We consider the extended NLS equation with third order dispersion:

$$\partial_t u - \partial_x^3 u + i\alpha \partial_x^2 u + i|u|^2 u = 0, \quad t \in [-T, T], \quad x \in \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}, \quad (1)$$

$$u(0, x) = u_0(x), \quad x \in \mathbf{T}, \quad (2)$$

where α is a real constant with $2\alpha/3 \notin \mathbf{Z}$ and $T > 0$. In (1), all the parameters are normalized except for α . Equation (1) appears as a mathematical model for nonlinear pulse propagation phenomena in various fields of physics, especially in nonlinear optics (see [54], [27] and [1]). So far, equation (1) without the third order derivative, that is, the cubic NLS equation has attracted much mathematical and physical interest. Recently, as the ultra-short pulse has become important in the photonic crystal fiber, an increasing attention among theoretical and experimental physicists in nonlinear optics has been paid to the role of the third order dispersion in equation (1).

From a viewpoint of the PDE theory, the well-posedness issue of the Cauchy problem for nonlinear evolution equations such as (1)-(2) is one of the most fundamental problems. The Cauchy problem is said to be locally (resp. globally) well-posed if the following three properties hold: (i) local (resp. global) existence of solutions, (ii) uniqueness of solutions, (iii) continuous dependence of solutions on initial data. We refer to the local and the global well-posedness as (LWP) and (GWP), respectively. The Cauchy problem is said to be ill-posed if it is not well-posed. Many mathematicians have been studying what is the largest space where the Cauchy problem of a nonlinear evolution equation at hand is well-posed. Especially, there has been a great progress in nonlinear dispersive equations for the last two decades. In this note, we take equation (1) as an example to explain recent results obtained by the author in collaboration with Nobu Kishimoto (RIMS, Kyoto University), Tomoyuki Miyaji (Meiji Inst. Adv. Stud. Math. Sci., Meiji University), Tadahiro Oh (The University of Edinburgh) and Nikolay Tzvetkov (University of Cergy-Pontoise).

First, in Section 2, we consider solving the Cauchy problem (1)-(2) in H^s for $s < 0$, elements of which may not be functions but distributions. When we work with the space consisting of distributions, the problem we immediately meet with is how the nonlinear term can make sense, because the product of distributions is not necessarily well-defined. In 1993, Bourgain [4] presented the so-called Fourier restriction method with the Fourier restriction norm. The Fourier restriction norm and its variants succeeded in capturing specific features of nonlinear oscillations and have been applied to many nonlinear evolution equations describing nonlinear wave phenomena (see, e.g.,

This work was supported by JSPS KAKENHI Grant Numbers 17H02853 and 16K13770.

2010 Mathematics Subject Classification: Primary 35Q55, 35Q53, Secondary 35A01, 35A02, 35A10.

Keywords: 3rd order NLS, well-posedness of the Cauchy problem, smoothing effect, Fourier restriction method, resonant frequency.

* e-mail: tsutsumi@math.kyoto-u.ac.jp

[4], [13], [18], [22]-[26], [32], [34]-[36], [40], [42], [44]-[49], [57]-[60]). The Fourier restriction method also led a new mathematical insight into the nonlinear interaction of multiple waves. Indeed, it revealed the smoothing type effect of the nonlinear dispersive equation, which helped us to investigate how to give an interpretation of the nonlinearity. The nonlinear dispersive equation *never* has such a smoothing effect as the nonlinear parabolic equation. However, the nonlinear interaction would often yield the smoothing type estimate, which would lead to (LWP) or (GLW) for the Cauchy problem of the corresponding nonlinear dispersive equation.

Next, we explain what we can analyze from the application of the smoothing type estimate in Section 2 to other problems. In Section 3, we consider the global attractor of the third order Lugiato-Lefever equation:

$$\partial_t u - \partial_x^3 u + i\alpha \partial_x^2 u + u + i|u|^2 u = f, \quad t > 0, \quad x \in \mathbf{T}. \quad (3)$$

The Lugiato-Lefever equation is nothing other than the nonlinear Schrödinger equation with damping and forcing. Roughly speaking, the proof for the existence of the global attractor consists of two steps. The first step is to prove the existence of the absorbing set, which follows from standard energy estimates. The second step is to prove the precompactness of orbits of solutions. In the case of nonlinear parabolic equations, the precompactness of orbits follows from the smoothing effect of the parabolic equation. But in the case of nonlinear dispersive equations, solutions to (3) themselves do not have extra regularity unlike parabolic equations. Nevertheless, we can show that the regular part of solutions to (3) remain bounded in t , while the nonregular part of solutions to (3) converges to zero as $t \rightarrow \infty$.

In Section 4, we consider the ill-posedness of the Cauchy problem for the nonlinear Schrödinger equation with third order dispersion and intrapulse Raman scattering term (see (2.3.43) on page 40 of [1]):

$$\begin{aligned} \partial_t u &= \alpha_1 \partial_x^3 u + i\alpha_2 \partial_x^2 u + i\gamma_1 |u|^2 u + \gamma_2 \partial_x (|u|^2 u) - i\Gamma u \partial_x (|u|^2), \\ t &\in [-T, T], \quad x \in \mathbf{T}, \end{aligned} \quad (4)$$

where α_j, γ_j ($j = 1, 2$) and Γ are real constants and T is a positive constant. The last term on the right side of (4) represents the effect of intrapulse Raman scattering. When the ultrashort pulse propagates along the optical fiber, the effect of Raman scattering is not negligible. A large number of numerical simulations for (4) have been made so far by researchers of optical physics (see, e.g., [1, Section 2.4.1] and [19]). As pointed out in [19] and [61], equation (4) is apt to lead to inaccuracies or even unphysical results in numerical simulations, while (4) has led to important physical insights into many different propagation effects because of its simple form. In a mathematically rigorous sense, we may say that the Raman scattering term gives rise to numerical inaccuracies. This is because the Cauchy problem of (4) is ill-posed, that is, not well-posed in H^s for any $s \geq 1$. The proof for the ill-posedness is based on the smoothing type effect of other nonlinear terms.

In Section 5, we consider the quasi-invariance of Gaussian measures for equation (1). In infinite dimension spaces, Gaussian or weighted Gaussian measures play a fundamental role. A typical example of weighted Gaussian measures is the so-called Gibbs measure, which is expected to be invariant under a flow generated by a system under consideration. In general, it is difficult to construct the Gibbs measure though it is natural and useful. Furthermore, the support of the Gibbs measure is in a less

regular space than the energy space. In other words, the set of all solutions in the energy space is measure zero with respect to the Gibbs measure and so the Gibbs measure can not capture the energy class solution. It is natural to ask how the Hamiltonian PDEs transport Gaussian measures, with respect to which the set of all solutions in the energy space is not measure zero. This kind of problem has been extensively studied from a viewpoint of the probability theory (see, e.g., Cameron and Martin [6], Kuo [38] and Ramer [55]). Recently, Tzvetkov [65] and Oh and Tzvetko [51, 52] have made a closer investigation into the transport of Gaussian measures under flows of nonlinear dispersive equations. For equation (1), this problem is studied in [50] by Ramer's theorem and the smoothing type effect.

This note is based on the author's plenary talk in the MSJ Autumn Meeting 2017, which was held at Yamagata University on September 11–14, 2017 (see [63, pages 13–26]). The author has slightly modified Sections 1-4 and created Section 5 to add new materials.

2. Local well-posedness in negative Sobolev spaces

In [41], it is shown that the Cauchy problem of (1)-(2) is globally well-posed in L^2 , which follows from the Strichartz estimate and the contraction argument together with the L^2 conservation. For $\lambda > 0$, we consider the scaling transformation: $u_\lambda(t, x) = \lambda^{3/2}u(\lambda^3t, \lambda x)$. If u is a solution of (1) and $\alpha = 0$, then u_λ satisfies equation (1) with \mathbf{T} and T replaced by $\mathbf{T}_\lambda = \mathbf{R}/2\pi\lambda^{-1}\mathbf{Z}$ and T/λ^3 , respectively. Furthermore, $\|D^s u_\lambda\|_{L^2(\mathbf{T}_\lambda)} = \lambda^{s+1}\|D^s u\|_{L^2(\mathbf{T})}$. Here and hereafter, we put $D = \mathcal{F}^{-1}|k|\mathcal{F}$, and \mathcal{F} and \mathcal{F}^{-1} denote the Fourier transform and the inverse Fourier transform, respectively. When $s = -1$, the quantity $\|D^s \cdot\|_{L^2(\mathbf{T}_\lambda)}$ is invariant under the scaling. The scaling suggests the borderline in regularity on whether a nonlinear evolution equation is well-posed or ill-posed. From the scaling point of view, it seems important and natural to study the well-posedness below L^2 , since the scaling critical exponent of (1) in the scale of Sobolev spaces H^s is $s = -1$. Therefore, one might expect the well-posedness of (1) and (2) in H^s for some $s < 0$. However, in [44], Molinet proves that the continuous dependence of solutions on initial data breaks down in the weak topology of L^2 for the cubic NLS equation. Furthermore, in [26], Guo and Oh prove the nonexistence of solution for the cubic NLS equation within the framework of H^s , $s < 0$. The proof in [44] is applicable to the Cauchy problem of (1) and (2) without any change, while Theorem 2.1 below and the nonexistence argument in [26] can be applied to the Cauchy problem of (1) and (2). Therefore, it is impossible to relax L^2 to bigger spaces H^s , $s < 0$ for the well-posedness of (1) and (2). Instead of (1), we consider the following reduced equation to which the resonance term breaking the well-posedness in H^s , $s < 0$ is removed:

$$\partial_t u - \partial_x^3 u + i\alpha \partial_x^2 u + i(|u|^2 - \frac{1}{\pi} \int_0^{2\pi} |u(t, x)|^2 dx)u = 0, \quad t \in [-T, T], \quad x \in \mathbf{T}. \quad (5)$$

Equation (5) follows from equation (1) by applying the gauge transformation:

$$v(t, x) = u(t, x)e^{\frac{i}{\pi}\|u_0\|_{L^2}^2 t} = u(t, x)e^{\frac{i}{\pi} \int_0^t \|u(s)\|_{L^2}^2 ds}. \quad (6)$$

Here, we note that both (1) and (5) have the L^2 norm conservation. It is known that the reduced equation is better than the original equation as far as the well-posedness issue is concerned (see, e.g., [4], [32], [10], [23] and [40]). If the solution u is fairly regular, for example, if the initial datum u_0 belongs to L^2 , then (5) is equivalent to

(1). Unless u_0 belongs to L^2 , (5) may be thought of as the equation resulting from the renormalization of the divergence of the L^2 norm.

In this section, we explain the nonlinear smoothing effect and state the time local well-posedness in $H^s(\mathbf{T})$ of the Cauchy problem (5) and (2) for $s > -1/6$ instead of the original equation (1). Before we state the main theorems in this section, we list notations which are used throughout this note. For any $a \in \mathbf{C}$, we put $\langle a \rangle = 1 + |a|$. Let $U(t) = e^{t(\partial_x^3 - i\alpha\partial_x^2)}$. Let \tilde{f} denote the Fourier transform of f in both the time and spatial variables. Let \hat{f} denote the Fourier transform of f only in the spatial variable x or only in the time variable t . For $b, s \in \mathbf{R}$, $T > 0$ and $u_0 \in \mathcal{D}'(\mathbf{T})$, we define the modified Fourier restriction norms $\|\cdot\|_{Z^{b,s}(u_0)}$ and $\|\cdot\|_{Z_T^{b,s}(u_0)}$ as follows. For $f \in \mathcal{S}'(\mathbf{R}^2)$ with $f(t, x) = f(t, x + 2\pi)$,

$$\|f\|_{Z^{s,b}(u_0)} = \left\{ \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \langle k \rangle^{2s} \langle \tau + k^3 - \alpha k^2 - |\hat{u}_0(k)|^2 \rangle^{2b} |\tilde{f}(\tau, k)|^2 d\tau \right\}^{1/2},$$

and for $f \in \mathcal{D}'((-T, T) \times \mathbf{T})$,

$$\|f\|_{Z_T^{s,b}(u_0)} = \inf \{ \|v\|_{Z^{s,b}(u_0)} \mid v \in Z^{s,b}(u_0), v(t) = f(t) \text{ on } (-T, T) \}.$$

We also define spaces $Z^{s,b}(u_0)$ and $Z_T^{s,b}(u_0)$ by the completions of $\mathcal{D}(\mathbf{R} \times \mathbf{T})$ and $C^\infty([-T, T] \times \mathbf{T})$ in the norms $\|\cdot\|_{Z^{s,b}(u_0)}$ and $\|\cdot\|_{Z_T^{s,b}(u_0)}$, respectively. For simplicity, when the dependence on u_0 does not need to be specified, spaces $Z^{s,b}(u_0)$ and $Z_T^{s,b}(u_0)$ are abbreviated to $Z^{s,b}$ and $Z_T^{s,b}$, respectively. When $u_0 = 0$, we write $Y^{s,b}$ and $Y_T^{s,b}$ for $Z^{s,b}(0)$ and $Z_T^{s,b}(0)$, respectively, which are the usual Fourier restriction spaces introduced by Bourgain [4]. We define the reduced nonlinearity F of (5) as follows.

$$\widehat{F(u)}(k) = i \sum_{\substack{k=k_1+k_2+k_3, \\ (k_1+k_2)(k_2+k_3) \neq 0}} \hat{u}(k_1)\hat{u}(k_2)\hat{u}(k_3) - i|\hat{u}(k)|^2\hat{u}(k). \quad (7)$$

We note that if u is a smooth function, $F(u)$ is equivalent to the fourth term on the left hand side of (5). We further set

$$\widehat{G(u)}(k) = \widehat{F(u)}(k) + i|\hat{u}_0(k)|^2\hat{u}(k),$$

which results from the elimination of the resonant effect from $F(u)$. Then, equation (5) can be written as the following integral equation.

$$\begin{aligned} \hat{u}(t, k) &= e^{-it(k^3 - \alpha k^2 + |\hat{u}_0(k)|^2)} \hat{u}_0(k) \\ &\quad - i \int_0^t e^{-i(t-r)(k^3 - \alpha k^2 + |\hat{u}_0(k)|^2)} \widehat{G(u)}(r, k) dr. \end{aligned} \quad (8)$$

Concerning the time local well-posedness of the Cauchy problem (5) and (2), we have the following theorem.

Theorem 2.1 (Miyaji-Y.T [42]). *Assume that $2\alpha/3 \notin \mathbf{Z}$. Let $s > -1/6$.*

(i) *For any $u_0 \in H^s$, there exists a positive constant T depending only on s and $\|u_0\|_{H^s}$ such that the Cauchy problem (5) and (2) has a unique solution u on $[-T, T]$ satisfying*

$$u \in C([-T, T]; H^s) \cap Z_T^{s,1/2}(u_0).$$

Furthermore, for any η with $0 < \eta < 1 + 6s$, the solution u satisfies

$$\sup_{t \in [-T, T]} \left(\sum_{k \in \mathbf{Z}} \langle k \rangle^{1+6s-\eta} \left| |\hat{u}(t, k)|^2 - |\hat{u}_0(k)|^2 \right| \right) < \infty, \quad (9)$$

$$G(u) \in Z_T^{s, -1/2}(u_0). \quad (10)$$

(ii) In addition to the assumptions of (i), let $\{u_{0n}\}$ be a sequence in H^s such that $u_{0n} \rightarrow u_0$ and let u and u_n be the solutions of (5) with $u(0) = u_0$ and $u_n(0) = u_{0n}$ given by part (i). Then, for any $0 < T' < T$,

$$u_n \longrightarrow u \text{ in } C([-T', T']; H^s) \quad (n \rightarrow \infty),$$

where T is the existence time of u given by part (i).

Remark 2.2. (i) In part (i) of Theorem 2.1, the property (9) implies a smoothing-type effect for the nonlinear term and the property (10) shows that the solution u satisfies equation (8) in $Z_T^{s, 1/2}$, by which equation (5) makes sense.

(ii) Theorem 2.1 also holds with the space $Z_T^{s, 1/2}(u_0)$ replaced by $Z_T^{s, b}(u_0)$ for b less than and close to $1/2$.

(iii) It is not known whether Theorem 2.1 holds for all $\alpha \in \mathbf{R}$. When $2\alpha/3 \in \mathbf{Z}$, it is open whether Lemma 2.5 below holds or not. The nonresonance condition like this was used by Takaoka [58] for the one dimensional Zakharov equations and by Oh [49] for the coupled system of KdV. Unless $\alpha/3 \notin \mathbf{Z}$, the nonresonance estimate (15) breaks down. This resonance seems to be a specific feature of the third order NLS equation (1) or (5), because it never occurs for the mKdV equation (see [4], [47] and [60]) and for the cubic NLS equation (see [25], [26] and [44]).

We now introduce new spaces to state the result on the nonuniqueness of solutions to the problem (5) and (2), which follows immediately from the argument by Shnirelman [56] and Christ [11]. In [56], Shnirelman presents a new idea for the construction of a weak nontrivial solution to the two dimensional incompressible Euler equation with zero initial data, which shows the nonuniqueness of solutions for the Euler equation. In [11], Christ employs the argument by Shnirelman [56] to show the nonuniqueness of solutions for the cubic NLS. By the $C^{-1}([-1, 1]; H^s)$, we denote the space $L^1((-1, 1); H^s)$ equipped with the norm

$$\|f\|_{C^{-1}([-1, 1]; H^s)} = \sup_{t \in [-1, 1]} \left\| \int_0^t e^{-t'(\partial_x^3 - i\alpha \partial_x^2)} f(t') dt' \right\|_{H^s}.$$

Let $\mathcal{C}^{-1}([-1, 1]; H^s)$ denote the completion of $C^{-1}([-1, 1]; H^s)$ in the above-mentioned norm. Let $\mathcal{L}(L^2, L^2)$ denote the space of all bounded linear operators from $L^2(\mathbf{T})$ to $L^2(\mathbf{T})$. By I we denote the identity operator from $L^2(\mathbf{T})$ to $L^2(\mathbf{T})$. In contrast to Theorem 2.1, we have the nonuniqueness of solutions in H^s , $s < 0$ unless they belong to such an auxiliary space as $Z_T^{s, 1/2}(u_0)$.

Theorem 2.3 (Miyaji-Y.T [42]). *Assume $s < 0$ and $3b + s > 0$. Then, there exists a nontrivial solution u of (5) and (2) with $u_0 = 0$ such that*

$$\begin{aligned} u &\in C([-1, 1]; H^s), \\ \exists \lim_{n \rightarrow \infty} F(\chi_n u) &\in \mathcal{C}^{-1}([-1, 1]; H^s), \end{aligned} \quad (11)$$

$$u \notin Z_1^{s, b}(0), \quad (12)$$

where $\{\chi_n\}$ is any sequence of Fourier multipliers on $L^2(\mathbf{T})$ such that each of their symbols has a finite support and $\chi_n \rightarrow I$ strongly in $\mathcal{L}(L^2, L^2)$.

Remark 2.4. (i) We first note that $F(u) = G(u)$ for the solution given by Theorem 2.3, since $u(0) = 0$. Therefore, the property (11) implies that

$$\exists \lim_{n \rightarrow \infty} G(\chi_n u) \in Y_1^{s,-1} (= Z_1^{s,-1}(0)),$$

while the solution u given by Theorem 2.1 satisfies the stronger condition (10). In [11], the solution u given by Theorem 2.3 is called a “weak solution in the extended sense” (see [11, Definitions 2.2 and 2.3 on page 3] for the precise definition of the solution). Namely, the nonlinearity is interpreted just in the limiting sense. On the one hand, such notion of solutions has been a great success in the field of stochastic nonlinear parabolic equations (see, e.g., Friz and Hairer [21, pages 221-222]). In this respect, this is by no means an ad hoc artificial notion. On the other hand, from the viewpoint of nonlinear partial differential equations, it might not be very satisfactory, because it does not provide a direct interpretation of the nonlinearity. Indeed, for some nonlinear evolution equations, this notion of solutions is not necessarily accepted (see, e.g., Arsénio and Saint-Raymond [2, lines 7-13 on page 359] for the measure-valued renormalized solution of the Vlasov-Maxwell-Boltzmann equations).

(ii) In Theorem 2.3, the assumption $3b + s > 0$ is not restrictive, because the nonuniqueness of solutions is interesting enough for a negative s close to 0.

(iii) The proof of Theorem 2.3 is exactly the same as that for the cubic NLS (see Christ [11]). We do not need to assume $2\alpha/3 \notin \mathbf{Z}$, because the linear dispersion plays no important role in the proof of Christ [11]. The nonuniqueness of solutions seems to be a feature of the negative Sobolev space rather than that of equation (5).

It is instructive to compare the well-posedness result of (5) with that of the reduced mKdV, since both the equations have the third order dispersion and the cubic nonlinearity. In [4], Bourgain proves (LWP) for the cubic NLS and the mKdV equations in L^2 and $H^{1/2}$, respectively, by using the Fourier restriction norm method. In [32], Kenig, Ponce and Vega refine the Fourier restriction norm method and improve the results on (LWP) for the KdV equation. For the mKdV equation on \mathbf{T} , the term $ik|\hat{u}(k)|^2\hat{u}(k)$ appearing in the reduced nonlinearity gives rise to the rapid oscillation of the phase of the solution (for equation (5), see formula (7)), which breaks down the uniform continuous dependence of solutions on initial data. For this reason, we need to modify the Fourier restriction norm so that we can get rid of this rapid oscillation factor. In [60], Takaoka and Tsutsumi introduce the modified Fourier restriction norm, which takes the rapid oscillation effect into account, and they succeed in bringing down the lower bound $s \geq 1/2$ to $s > 3/8$ for (LWP) in H^s by using a kind of smoothing effect. In [47], Nakanishi, Takaoka and Tsutsumi further improve the result on the smoothing effect to show (LWP) of mKdV in H^s for $s > 1/3$. In [47], they also prove the existence of solutions for the reduced mKdV in H^s , $s > 1/4$ without the uniqueness of solutions. In [48], Nguyen shows the existence of power series solutions for the reduced mKdV by using the argument of Christ [10] in $\mathcal{FL}^{1/2,p}$, $p > 4$. We note that $\mathcal{FL}^{1/2,4}$ is scaling equivalent to $H^{1/4}$. Recently, in [46], Molinet, Pilod and Vento have showed the unconditional uniqueness in H^s , $s \geq 1/3$ as well as (LWP) of mKdV in H^s , $s = 1/3$ (see also Kwon and Oh [40] for the unconditional uniqueness of mKdV in H^s , $s \geq 1/2$ and Guo, Kwon and Oh [25] for the unconditional uniqueness of the cubic NLS in H^s , $s \geq 1/6$). In [46], Molinet, Pilod and Vento have made a close investigation into the

a priori bound and the smoothing effect by using a kind of the short time Fourier restriction norm argument (see [46, Lemma 3.10 and Theorem 4.1]).

From a scaling point of view, the difference in Sobolev exponent between (1) and mKdV is $1/2$. For this reason, we may expect that (LWP) holds in $1/2$ less regular Sobolev spaces for (5) than for the reduced mKdV. In this respect, Theorem 2.1 seems to be natural when we compare it with the result on (LWP) for the reduced mKdV equation. However, the uniqueness of solutions breaks down in H^s , $s < 0$ for (5) without the auxiliary space $Z_T^{s,b}(u_0)$ (see Theorem 2.3), while it holds in H^s , $s \geq 1/3$ for the reduced mKdV without any auxiliary spaces (see Molinet, Pilod and Vento [46]).

The following lemma implies the smoothing effect for the nonlinear term, which plays a crucial role in the proof of Theorem 2.1.

Lemma 2.5. *Assume $2\alpha/3 \notin \mathbf{Z}$ and $0 > s > -1/4$. Let $u_0 \in C^\infty(\mathbf{T})$, let u be the global smooth solution of (5) with $u(0) = u_0$ and let $T > 0$. Then, there exists $1/2 > b_0 > 0$ with the following properties: For any b and η with $1/2 > b \geq b_0$ and $\eta > 0$, there exists a positive constant C such that*

$$\begin{aligned} & \sum_{k \in \mathbf{Z}} \langle k \rangle^{1+6s-\eta} \left| |\hat{u}(t, k)|^2 - |\hat{u}_0(k)|^2 \right| \\ & \leq C \left[\|u(t)\|_{H^{3s/2}}^4 + \|u_0\|_{H^{3s/2}}^4 + \|u\|_{Z_T^{s,b}(u_0)}^6 \right], \quad t \in [-T, T], \end{aligned}$$

where C depends only on s , η and $\|u_0\|_{H^s}$.

Remark 2.6. Let $0 > s > -1/4$. Lemma 2.5 implies the smoothing effect, since $(1 + 6s - \eta)/2 > s > 3s/2$ for sufficiently small $\eta > 0$, that is, the order of derivative on the right hand side is smaller than that on the left hand side of the inequality. On the one hand, it would be helpful for the construction of a solution in H^s , $s > -1/4$ without the uniqueness (see [47, Theorem 1.1 on page 1637] for the reduced mKdV and [13], [26], [35] and [36] for the cubic NLS). On the other hand, it would not be sufficient for the uniqueness of solutions when $-1/6 > s > -1/4$, because $1 + 6s < 0$. This kind of smoothing effect seems to be the only way we know to treat such resonant terms as $i|\hat{u}(t, k)|^2 \hat{u}(t, k)$ when we estimate the difference of two solutions (see, e.g., a remark before Theorem 4.1 in [46]).

We take the Fourier transform in x of (5) to have

$$\begin{aligned} & \partial_t \hat{u}(t, k) + i(k^3 - \alpha k^2) \hat{u}(t, k) \\ & = -i \sum_{\substack{k=k_1+k_2+k_3, \\ (k_1+k_2)(k_2+k_3) \neq 0}} \hat{u}(t, k_1) \hat{u}(t, k_2) \hat{u}(t, k_3) + i|\hat{u}(t, k)|^2 \hat{u}(t, k). \end{aligned} \tag{13}$$

For $k = k_1 + k_2 + k_3$, we define a phase function Φ as follows

$$\begin{aligned} \Phi(k, k_1, k_2, k_3) &= (\tau + k^3 - \alpha k^2) - (\tau_1 + k_1^3 - \alpha k_1^2) \\ & \quad - (\tau_2 + k_2^3 + \alpha k_2^2) - (\tau_3 + k_3^3 - \alpha k_3^2) \\ &= 3(k_1 + k_2)(k_2 + k_3)(k_3 + k_1 - 2\alpha/3), \end{aligned} \tag{14}$$

where $\tau = \tau_1 + \tau_2 + \tau_3$. The identity (14) represents the nonlinear interaction making three input waves into one output wave, which is often called ‘‘four-wave mixing’’ in

the physics literature. We set $L = \max\{|k_1 + k_2|, |k_2 + k_3|, |k_3 + k_1 - 2\alpha/2|\}$. By the identity (14), for $k \in \mathbf{Z}$ with $|k|$ large, we have

$$\exists c > 0; |\Phi| \geq \begin{cases} c|k||k_1 + k_2||k_2 + k_3|, & |k_3 + k_1 - 2\alpha/3| \geq L, \\ c|k||k_2 + k_3||k_3 + k_1 - 2\alpha/3|, & |k_1 + k_2| \geq L, \\ c|k||k_1 + k_2||k_3 + k_1 - 2\alpha/3|, & |k_2 + k_3| \geq L. \end{cases} \quad (15)$$

Pairs of Fourier frequencies (k_1, k_2, k_3) with $\Phi = 0$ in (14) are called resonant frequencies.

We now illustrate an idea of the proof of Lemma 2.5. We use the interaction representation $\hat{v}(t, k) = e^{it(k^3 - \alpha k^2)} \hat{u}(t, k)$. We differentiate $|\hat{v}(t, k)|^2$ in t by using (13) and integrate the real part of the resulting equation in t to have

$$\begin{aligned} |\hat{v}(t, k)|^2 - |\hat{u}_0(k)|^2 &= 2\text{Im} \int_0^t \sum_{\substack{k=k_1+k_2+k_3 \\ (k_1+k_2)(k_2+k_3) \neq 0}} e^{ir\Phi(k, k_1, k_2, k_3)} \\ &\quad \times \hat{v}(r, k_1) \hat{v}(r, k_2) \hat{v}(r, k_3) \bar{\hat{v}}(r, k) \, dr, \quad t \in \mathbf{R}. \end{aligned} \quad (16)$$

Integration by parts and the fact that $|\hat{u}| = |\hat{v}|$ yield

$$\begin{aligned} |\hat{u}(t, k)|^2 - |\hat{u}_0(k)|^2 &= 2\text{Im} \left[\sum_{\substack{k=k_1+k_2+k_3 \\ (k_1+k_2)(k_2+k_3) \neq 0}} (i\Phi)^{-1} \right. \\ &\quad \times \left(\hat{u}(t, k_1) \hat{u}(t, k_2) \hat{u}(t, k_3) \bar{\hat{u}}(t, k) - \hat{u}_0(k_1) \hat{u}_0(k_2) \hat{u}_0(k_3) \bar{\hat{u}}_0(k) \right) \\ &\quad - \int_0^t \sum_{\substack{k=k_1+k_2+k_3 \\ (k_1+k_2)(k_2+k_3) \neq 0}} (i\Phi)^{-1} \\ &\quad \times \left\{ \left(\sum_{\substack{k_1=k_{11}+k_{12}+k_{13} \\ (k_{11}+k_{12})(k_{12}+k_{13}) \neq 0}} \hat{u}(r, k_{11}) \hat{u}(r, k_{12}) \hat{u}(r, k_{13}) \right) - |\hat{u}(r, k_1)|^2 \hat{u}(r, k_1) \right\} \\ &\quad \times \hat{u}(r, k_2) \hat{u}(r, k_3) \bar{\hat{u}}(r, k) \, dr \left. \right] + \text{other similar terms}. \end{aligned} \quad (17)$$

The factor Φ^{-1} and (15) yield an extra gain of derivatives on the right side of (17) (for the details, see Miyaji and Tsutsumi [42, Proof of Lemma 2.4]).

Open Problem 1. (i) In Theorem 2.1, we proved (LWP) in H^s , $s > -1/6$ for the Cauchy problem of (5). The following question naturally arises: Does the local solution exist globally in time? There are several papers concerning this question posed in related equations. In the case of \mathbf{R} , Koch and Tataru [36] proved the global existence of solution for the Cauchy problem of the cubic NLS in H^s , $s \geq -1/4$ without uniqueness. They combined the short time $X^{s,b}$ space by Ionescu and Kenig [29] and the I-method by Colliander, Keel, Staffilani, Takaoka and Tao [14] to show the *a priori* bound needed for the proof of the global existence. The argument using the short time $X^{s,b}$ space is compatible with the I-method, while it seems difficult to adjust the I-method to the $Z^{s,b}(u_0)$ space. To what nonlinear dispersive equations can the argument combining the short time $X^{s,b}$ space and the I-method be applied for (GWP) in low regularity spaces?

(ii) Following the argument by Koch and Tataru [36], in the case of \mathbf{T} , Oh and Wang [53] have recently proved not only the global existence of solutions for the Cauchy

problems of the reduced fourth order NLS in H^s , $s \geq -1/3$ but also the uniqueness of solutions and the continuous dependence of solutions on initial data by using the smoothing effect similar to (9). When the cubic resonance term $|\hat{u}(t, k)|^2 \hat{u}(t, k)$ appears, the only proof available for the uniqueness is to use such smoothing type estimates as (9). To what nonlinear dispersive equations can the argument using the smoothing estimate of type (9) can be applied for (GWP) in low regularity spaces? In [13], for example, Christ, Colliander and Tao proved the following smoothing type estimate for the cubic NLS on \mathbf{R} .

$$\|u\|_{C([-T, T]; H^s)}^2 - \|u_0\|_{H^s}^2 \leq C \|u\|_{X_T^{r, b}}^4, \quad 0 > s > -1/2, \quad r > -1/4, \quad b > 1/2,$$

where $X_T^{r, b}$ is the corresponding Fourier restriction space on \mathbf{R} , being defined as $Y_T^{s, b}$ with \mathbf{T} replaced by \mathbf{R} . We note that this is weaker than (9). It is not sufficient for the uniqueness of solutions while it ensures the local existence of solutions. Can we show a stronger smoothing estimate as in Lemma 2.5 for the cubic NLS on \mathbf{R} ?

3. Global attractor of 3rd order Lugiato-Lefever equation

Without damping and forcing the solution u of (1)-(2) formally satisfies the following three conservations, that is, the mass, the momentum and the energy conservations for $t > 0$ (see [54, lines 7 to 10 on page 2326]).

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad (18)$$

$$\text{Im}(\partial_x u(t), u(t)) = \text{Im}(\partial_x u_0, u_0), \quad (19)$$

$$\begin{aligned} \text{Im}(\partial_x^2 u(t), \partial_x u(t)) + \alpha \|\partial_x u(t)\|_{L^2}^2 - \frac{1}{2} \|u(t)\|_{L^4}^4 \\ = \text{Im}(\partial_x^2 u_0, \partial_x u_0) + \alpha \|\partial_x u_0\|_{L^2}^2 - \frac{1}{2} \|u_0\|_{L^4}^4, \end{aligned} \quad (20)$$

where (\cdot, \cdot) denotes the scalar product of $L^2(\mathbf{T})$. The energy functional defined as in (20) is neither positive definite nor negative definite, because it includes the L^2 scalar product of the second and the first derivatives of the solution. This suggests that the energy is not useful for controlling the global behavior of the solution. Therefore, we need to consider the global solution in L^2 and as a result, we need to construct the global attractor in L^2 instead of the H^1 global attractor. The construction of global attractor in L^2 causes a serious problem on the precompactness of orbits.

Let $(X, \|\cdot\|)$ be a Banach space and let $S : X \times [0, \infty) \rightarrow X$ be a semiflow (continuous mapping from $X \times [0, \infty)$ to X with $S(t+s) = S(t)S(s)$, $t, s \geq 0$ and $S(0) = I$, where I is the identity operator). Let $A \subset X$ be a compact set. The set A is said to be global (or universal) attractor for S if A satisfies

$$\begin{aligned} (i) (\text{invariant}) \quad S(t)A = A, \quad t > 0, \\ (ii) (\text{unifrom attraction}) \quad \forall D \subset X, \text{ bounded} \implies \\ d(S(t)D, A) \rightarrow 0 \quad (t \rightarrow \infty), \end{aligned}$$

where $d(S(t)D, A) = \sup_{v \in D} \inf_{w \in A} \|S(t)v - w\|$.

Remark 3.1. The global attractor characterizes the asymptotic behavior of all global solutions for the nonlinear evolution equation with dissipation, if it exists.

Theorem 3.2 (Miyaji-Y.T [41]). *Assume $2\alpha/3 \notin \mathbf{Z}$ and $f \in L^2(\mathbf{T})$. Let $S : (u_0, t) \mapsto u(t)$ be the solution semiflow associated with (3) and (2). Then, there exists the unique global attractor $\mathcal{A} \subset L^2$ for S .*

The strategy of a proof for the global attractor is as follows:

(Step 1) Prove the existence of the absorbing set in L^2 . (It is *easy*, because this follows the L^2 energy inequality.)

(Step 2) Prove the precompactness in L^2 of orbits of solutions. (We have to investigate the regularity of the Duhamel term, which is our main task.)

Remark 3.3. (i) The methods to prove the precompactness of orbits of solutions can be classified into the following two groups:

(a) Use the smoothing effect, which is useful for nonlinear parabolic equations.

(b) Construct the global attractor in the weak topology and then show the convergence by attraction in the strong topology (it is due to John M. Ball [3] and improved by Molinet [45]).

The approach (b) is applicable to some perturbed Hamiltonian systems without smoothing effect. Indeed, Molinet's argument is applicable to (3), but our proof yields the following stronger result than the existence of global attractor: There exists $s > 0$ such that $\mathcal{A} \subset H^s$ and all solutions u of (3) with $u(0) = u_0 \in L^2$ satisfy

$$\begin{aligned} u(t) - V(t)u_0 &\in H^s, \\ \inf_{w \in \mathcal{A}} \|(u(t) - V(t)u_0) - w\|_{H^s} &\rightarrow 0 \quad (t \rightarrow \infty), \\ V(t)u_0 &= \sum_{k=-\infty}^{\infty} e^{-t(i(k^3 - \alpha k^2) + 1) + ikx} e^{-i \int_0^t \|u(s)\|_{L^2}^2 ds + i \int_0^t |\hat{u}(s,k)|^2 ds} \hat{u}_0(k). \end{aligned}$$

This implies that the smooth part of the solution remains, while the nonsmooth part exponentially decays to zero as $t \rightarrow \infty$. In [16] and [17], Erdoğan and Tzirakis use the smoothing effect of the Duhamel term to construct the global attractor for the KdV and the Zakharov equations. However, the whole Duhamel term can not become more regular than the initial datum in the case of the third order Lugiato-Lefever equation (3), which is in sharp contrast to the KdV and the Zakharov equations.

(ii) The linear damped KdV-Schrödinger evolution operator $U(t) = e^{t((\partial_x^3 - i\alpha \partial_x^2) - 1)}$ has no smoothing effect: If $h(t)$ belongs to $C([0, T]; H^s(\mathbf{T}))$, then, in general,

$$\int_0^t U(t-t')h(t') dt' \notin H^{s_1}, \quad s_1 > s.$$

We take the Fourier transform of the resulting equation from the gauge transformation of (3) by (6) to have the following equation.

$$\begin{aligned} \partial_t \hat{v}(t, k) + (i(k^3 - \alpha k^2) + 1) \hat{v}(t, k) & \tag{21} \\ + i \sum_{\substack{k_1 + k_2 + k_3 = k \\ (k_1 + k_2)(k_2 + k_3) \neq 0}} \hat{v}(t, k_1) \hat{v}(t, k_2) \hat{v}(t, k_3) - i |\hat{v}(t, k)|^2 v(t, k) \\ & = \hat{f}(k) e^{-\frac{i}{\pi} \int_0^t \|v(s)\|_{L^2}^2 ds}, \quad t > 0. \end{aligned}$$

The last term on the left side of (21) has no smoothing effect, but the last but one has no resonant frequencies, which leads to the smoothing effect. We remove the last term on the left side by the following gauge transformation:

$$\hat{w}(t, k) = e^{-i \int_0^t |\hat{v}(s,k)|^2 ds} \hat{v}(t, k).$$

Consequently, all what we have to do is to estimate a variant of the Fourier restriction norm with modification factor $e^{-i \int_0^t |\hat{v}(s,k)|^2 ds}$. The argument mentioned in Section 2 enables us to estimate it (for the details, see Miyaji and Tsutsumi [41]).

Open Problem 2. It is presumed that if $2\alpha/3 \in \mathbf{Z}$, the global attractor may exist. In this case, does equation (3) or (21) have a smoothing type effect similar to the case $2\alpha/3 \notin \mathbf{Z}$?

4. Ill-posedness of 3rd order NLS with Raman scattering term

In this section, we explain how the Raman scattering term gives rise to the ill-posedness for the Cauchy problem of (4). We first note that the L^2 norm is conserved for (4). Inspecting the Raman scattering term in the Fourier frequency space, the equation (4) can be rewritten as follows:

$$\begin{aligned} \partial_t u + ia\partial_x u &= \alpha_1 \partial_x^3 u + i\alpha_2 \partial_x^2 u + i\gamma_1 |u|^2 u + i\gamma_2 \partial_x (|u|^2 u) \\ &+ \frac{\Gamma}{(2\pi)^{3/2}} \sum_{k \in \mathbf{Z}} e^{-ikx} \sum_{(k_1+k_2)(k_2+k_3) \neq 0} (k_1 + k_2) \hat{u}(k_1) \hat{u}(k_2) \hat{u}(k_3) \\ &- \frac{\Gamma}{2\pi} \left(\sum_{k_2 \in \mathbf{Z}} k_2 |\hat{u}(k_2)|^2 \right) u, \quad t \in [-T, T], \quad x \in \mathbf{T}, \end{aligned} \quad (22)$$

where

$$a = \frac{\Gamma}{2\pi} \|u_0\|_{L^2}^2.$$

On the left side of (22), the Cauchy-Riemann type elliptic operator $\partial_t + ia\partial_x$ appears due to the Raman scattering term. For the proof of the ill-posedness, we need to show that the instability coming from the Cauchy-Riemann elliptic operator is dominant over the singularity caused by the nonlinear terms, which excludes a possibility that these two effects cancel out. Indeed, the nonlinear terms on the right side of (22) can be controlled in H^s for $s \geq 1$ by virtue of the smoothing type effect mentioned in Section 2 (see also [18], [25], [26] and [44] for the NLS equation, [40], [46], [47] and [60] for the mKdV equation, and [41, 42] for the third order NLS).

Thus, we have the following theorem concerning the non-existence of solution for the Cauchy problem of (22).

Theorem 4.1 (Kishimoto-Y.T [34]). *We assume $2\alpha_2/3\alpha_1 \notin \mathbf{Z}$. Let real numbers s, s_1 satisfy*

$$1 \leq s_1 \leq s < s_1 + 1.$$

Then, there exists $u_0 \in H^s(\mathbf{T})$ such that for any $T > 0$ the Cauchy problem (4) and (2) has no solution $u \in C([0, T]; H^{s_1}(\mathbf{T}))$ on $[0, T)$, nor solution $u \in C((-T, 0]; H^{s_1}(\mathbf{T}))$ on $(-T, 0]$.

Remark 4.2. (i) In the case of \mathbf{R}^n , the Cauchy problem of the semilinear Schrödinger equation is well-posed in regular Sobolev spaces (see Hayashi and Ozawa [28] and Chihara [7] for the one dimensional case and see Chihara [8] for the higher dimensional case). It is in sharp contrast to our case of \mathbf{T} . The difference between the cases of \mathbf{R} and \mathbf{T} is that the spectrum of the Laplacian is continuous in the former case, while it is discrete in the latter case.

(ii) The same nonexistence result as Theorem 4.1 holds for the equation (4) with $\alpha_1 = 0$ (see Kishimoto and Tsutsumi [34, Proposition 2.5]).

(iii) Tsugawa [62] introduced the notion of “parabolic smoothing effect”. This is a smoothing effect of parabolic type, which the nonlinear interaction yields. This might be helpful for the study of the ill-posedness.

We also have the following theorem concerning the norm inflation for the Cauchy problem of (22).

Theorem 4.3. *We assume that $2\alpha_2/3\alpha_1 \notin \mathbf{Z}$ holds. Let real numbers s, s_1 satisfy*

$$1 \leq s_1 \leq s < \min\left\{\frac{5}{3}s_1 + \frac{1}{2}, s_1 + \frac{3}{2}\right\}. \quad (23)$$

Then, for any $\varepsilon, \tau > 0$ there exists a real analytic function $\psi_{\varepsilon, \tau}$ satisfying $\|\psi_{\varepsilon, \tau}\|_{H^s} \leq \varepsilon$ such that if there exists a solution $u \in C([0, \tau]; H^{s_1}(\mathbf{T}))$ to (22) with the initial condition $u(0) = \psi_{\varepsilon, \tau}$, it holds that

$$\sup_{t \in [0, \tau]} \|u(t)\|_{H^{s_1}} \geq \varepsilon^{-1}.$$

The same is true for the negative time direction.

There are many papers concerning the well-posedness issue for the Cauchy problem of nonlinear dispersive equations (see, e.g., [4], [11], [12], [18], [25], [26], [32, 33], [37], [40], [41, 42], [44], [46], [47], [60] and [64]). For the well-posedness of linear Schrödinger equations, Mizohata [43] and Chihara [9] studied necessary and sufficient conditions in the cases of \mathbf{R}^n and \mathbf{T}^n , respectively. In [9], Chihara also treated the ill-posedness of the nonlinear Schrödinger equation. These works on linear equations give deep insight to nonlinear dispersive equations. On the other hand, in the nonlinear case, a linearized equation can not determine all properties of the original nonlinear equation, because a possible singularity caused by the nonlinearity might cancel out the one coming from the instability of the linearized operator.

In spite of the negative theorem mentioned above, there have been a large number of papers on numerical simulations of (4) (see, e.g., [1]). Those numerical computations suggest the following two things (see Erkontalo, Genty, Wetzel and Dudley [19]): First, (4) is apt to lead to numerical inaccuracies. Second, the instability leading to numerical inaccuracies seems to account for some physically interesting phenomena. In fact, in most of those papers, the hyperbolic secant and the Gaussian pulses are chosen as initial data, which are analytic functions. This observation makes us apply the Cauchy-Kowalevsky type theorem to the Cauchy problem of (4). We now describe the solvability for the Cauchy problem of (22) in the analytic function space. We begin with the definition of the function space with which we work.

Definition 4.4. For $r > 0$, we define a Banach space $\mathcal{A}(r)$ by

$$\mathcal{A}(r) := \{f \in L^2(\mathbf{T}) \mid \|f\|_{\mathcal{A}(r)} := \|e^{r|k|} \hat{f}(k)\|_{\ell^1(\mathbf{Z})} < \infty\}.$$

Remark 4.5. The function space $\mathcal{A}(r)$ was essentially introduced by Ukai [66, norm (2.6) and Definition 2.2 on page 143] for the Boltzmann equation, Kato and Masuda [30, the definition of $A(r)$ on page 459] for a class of nonlinear evolution equations and by Foias and Temam [20, (1.10) on page 361] for Navier-Stokes equations. Functions in $\mathcal{A}(r)$ are real analytic and have analytic extensions on the strip $\{z \in \mathbf{C} \mid |\Im z| < r\}$ (see, e.g., [31, Exercise 4.4 on page 28]).

Proposition 4.6 (Kishimoto-Y.T [34]). *Let $\alpha_j, j = 1, 2$ be two real numbers and let $r > 0$. For any $u_0 \in \mathcal{A}(r)$, there exists $T > 0$ such that the Cauchy problem (4)–(2) has a unique solution $u \in C([-T, T]; \mathcal{A}(r/2))$ on $(-T, T)$. Moreover, T can be chosen as*

$$T \gtrsim \min\{1, r\} \|u_0\|_{\mathcal{A}(r)}^{-2},$$

where the implicit constant does not depend on r and u_0 .

Remark 4.7. We do not have to assume the nonresonance condition $2\alpha_2/3\alpha_1 \notin \mathbf{Z}$ in Proposition 4.6. Even when $\alpha_1 = \alpha_2 = 0$, Proposition 4.6 holds.

Example 4.8. Consider as initial data the rescaled periodic Gaussian g_λ ($\lambda > 0$) defined by

$$\hat{g}_\lambda(k) := \lambda e^{-\lambda^2 k^2}, \quad k \in \mathbf{Z}.$$

We choose $r = \lambda$ and estimate the $\mathcal{A}(\lambda)$ -norm of g_λ as

$$\|g_\lambda\|_{\mathcal{A}(\lambda)} \lesssim \int_0^\infty \lambda e^{-\lambda^2 \xi^2 + \lambda \xi} d\xi + \sup_{\xi \geq 0} \lambda e^{-\lambda^2 \xi^2 + \lambda \xi} \lesssim 1 + \lambda.$$

Proposition 4.6 then shows that if $0 < \lambda \lesssim 1$, the corresponding solution u_λ to (4) exists on $(-T_\lambda, T_\lambda)$ with

$$T_\lambda \gtrsim \lambda. \quad (24)$$

In most literature, numerical computations are carried out for five to ten times as long a period of time as the dispersion length (see, e.g., [1, Figure 4.23 on page 112]). When $\alpha_1 = 0$ and the initial datum is the rescaled periodic Gaussian pulse defined as above, the dispersion length L_D is defined as $L_D = \lambda^2/|\alpha_2|$ (see [1, (4.4.2) in Section 4.4]). From (24), it is presumed that the numerical solution for the ill-posed Cauchy problem (4) and (2) may approximate the analytic solution given by Proposition 4.6 for as long a period of time as the length of λ ($= \frac{|\alpha_2|}{\lambda} L_D$). We note that if λ is small and $|\alpha_2| \sim 1$, this time range may be able to cover the period of time for which the numerical simulations are carried out in previous literature.

Open Problem 3. If the initial datum is a hyperbolic secant, a Gaussian or a super-Gaussian pulse, does the solution to (4) exist globally in time?

5. Quasi-invariant Gaussian measures for 3rd order NLS

For $s > \frac{1}{2}$, let μ_s be the mean-zero Gaussian measure on $L^2(\mathbf{T})$ with the covariance operator $2(\text{Id} - \Delta)^{-s}$, formally written as

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = \prod_{n \in \mathbf{Z}} Z_{s,n}^{-1} e^{-\frac{1}{2}\langle n \rangle^{2s} |\hat{u}_n|^2} d\hat{u}_n. \quad (25)$$

More specifically, we can define μ_s as the induced probability measure under the following map

$$\omega \in \Omega \mapsto u^\omega(x) = u(x; \omega) = \sum_{n \in \mathbf{Z}} \frac{g_n(\omega)}{\langle n \rangle^s} e^{inx}, \quad (26)$$

where $\langle \cdot, \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ and $\{g_n\}_{n \in \mathbf{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables (i.e. $\text{Var}(g_n) = 2$) on a probability space (Ω, \mathcal{F}, P) . It is easy to see that u^ω in (26) lies in $H^\sigma(\mathbf{T}) \setminus H^{s-\frac{1}{2}}(\mathbf{T})$ for $\sigma < s - \frac{1}{2}$, almost surely. Namely, μ_s is a Gaussian probability measure on $H^\sigma(\mathbf{T})$, $\sigma < s - \frac{1}{2}$. Moreover, for the same range of σ , the triplet (H^s, H^σ, μ_s) forms an abstract Wiener space (see, e.g., Kuo [39] and Daleckey and Formin [15]).

The aim in this section is to study the transport property of the Gaussian measures μ_s on Sobolev spaces under the dynamics of (1). We first recall the following definition of quasi-invariant measures. Given a measure space (X, μ) , we say that μ is *quasi-invariant* under a transformation $T : X \rightarrow X$ if the transported measure $T_*\mu = \mu \circ T^{-1}$ and μ are equivalent, i.e. mutually absolutely continuous with respect to each other.

We now state our main result.

Theorem 5.1 (Oh-Y.T-Tzvetkov [50]). *Let $\frac{2}{3}\alpha \notin \mathbf{Z}$. Then, for $s > \frac{3}{4}$, the Gaussian measure μ_s is quasi-invariant under the flow of the cubic 3rd order NLS (1).*

For a Hilbert space H , let $GL(H)$ be the group of invertible bounded linear operators from H to H and let $\mathcal{HS}(H)$ be the space of Hilbert-Schmidt operators from H to H . The Jacobi theorem for nonlinear transformations of Gaussian measures was showed by Ramer [55] in 1974.

Theorem 5.2 (Ramer [55]). *Let (i, H, E) be an abstract Wiener space and μ be the standard Gaussian measure on E with variance parameter 1. Let U be an open subset of E and $T = I + K : U \rightarrow E$ a continuous nonlinear transformation of U such that*

- (1) *T is a homeomorphism of U onto an open subset of E ,*
- (2) *$K(U) \subset H$ and The resulting map $K : U \rightarrow H$ is continuous,*
- (3) *K is an H - C^1 map; its H -derivative at x , $DK(x)$ is a Hilbert-Schmidt operator for each $x \in U$, the resulting map $DK : U \rightarrow \mathcal{HS}(H)$ is continuous, and $I_H + DK(x) \in GL(H)$ for each $x \in U$.*

Then μ and μT are mutually absolutely continuous as measures on U .

Remark 5.3. (i) The definition of H - C^1 class is different from the standard C^1 class (for the precise definition, see [38, DEFINITION on page 172]). Roughly speaking, the H - C^1 class consists of all functions on E which are continuously differentiable in the “ H -element” direction. For functions on the abstract Wiener space E , the differentiability in the H -element direction is important.

(ii) When the Hilbert-Schmidt operator is replaced by the nuclear operator in the assumption (3), Theorem 5.2 is proved by Cameron and Martin [6] for the calssical Wiener space and by Kuo [38] for the abstract Wiener space.

Recently, by using the PDE techniques, Tzvetkov [65] and Oh and Tzvetkov [51, 52] have applied Theorem 5.2 to nonlinear transformations associated with nonlinear dispersive equations. Following their argument, we apply Theorem 5.2 to the transformation T associated with (5): for fixed $t \in \mathbf{R}$,

$$Tu_0 = u_0 - i \int_0^t S(-t')F(u(t')) dt', \quad (27)$$

where $S(t) = e^{t(\partial_x^3 - i\alpha\partial_x^2)}$, u is a solution of (5) with $u(0) = u_0$ and $F(u)$ is defined as in (7). In our case, we define the operator K as the second term on the right side of (27). Let $\eta > 0$ be sufficiently small. We have only to show that K and DK are mappings from H^s to $H^{s+1/2+\eta}$ and from $H^{s+1/2+\eta}$ to $H^{s+1+2\eta}$ for initial datum u_0 in H^s , which implies that $K(E) \subset H$ and $DK \in \mathcal{HS}(H)$ with $H = H^{s+1/2+\eta}$ and $E = H^s$. This assertion follows from such a smoothing type estimate as (9) in Theorem 2.1.

Open Problem 4. (i) What can we learn from the quasi-invariance of Gaussian measures? For example, the Gibbs measure enables us to prove the global existence of solutions for initial data in the low regularity space which is the support of the Gibbs measure (see, e.g., [5]). Is the quasi-invariance of Gaussian measures helpful for the study of global existence and the asymptotic behavior of solutions?

(ii) The space-time white noise belongs to $H^{-d/2-}([0, \infty) \times \mathbf{T}^d)$. Therefore, noises are usually regarded as irregular perturbations from a viewpoint of the PDE theory. It is very interesting to ask whether the Cauchy problem of the nonlinear dispersive equation with additive or multiplicative noise is well-posed in suitable function spaces (see, e.g., [21] and references therein).

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